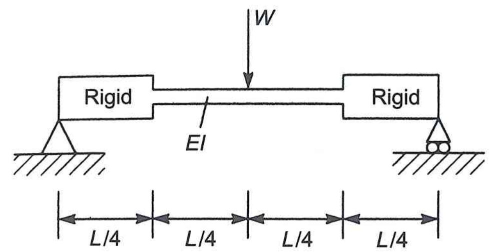


FIGURE P.13.10

P.13.11 Calculate the horizontal and vertical components of the deflection at the centre of the simply supported span AB of the thick Z-section beam shown in Fig. P.13.11. Take $E = 200\ 000\ \text{N/mm}^2$.

Ans. $u = 2.45$ mm (to right), $v = 1.78$ mm (upwards).

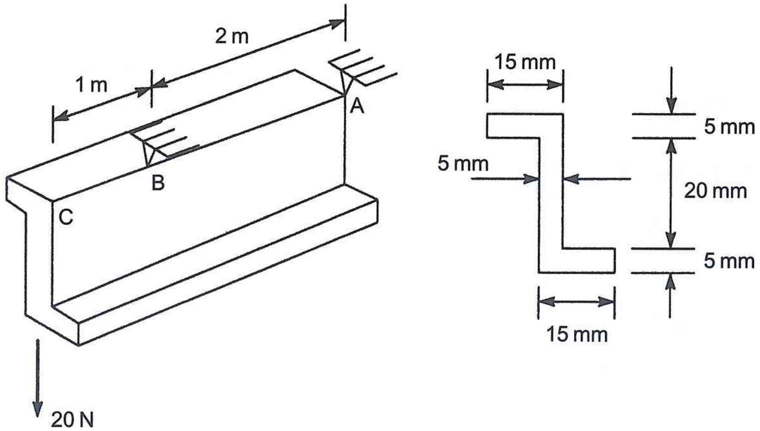


FIGURE P.13.11

P.13.12 A cantilever beam of length 5 m has the cross section shown in Fig. P.13.12. If the beam carries a vertically downward uniformly distributed load of intensity 10 kN/m calculate the magnitude and direction of the deflection of the free end of the beam. Young's modulus $E = 15000\ \text{N/mm}^2$.

Ans. 4.6 mm at 24.3° to the right of vertical.

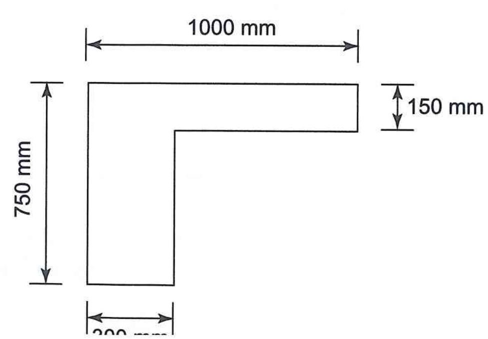


FIGURE P.13.12

P.13.13 The simply supported beam shown in Fig. P.13.13 supports a uniformly distributed load of 10 N/mm in the plane of its horizontal flange. The properties of its cross section referred to horizontal and vertical axes through its centroid are $I_z = 1.67 \times 10^6\ \text{mm}^4$, $I_y = 0.95 \times 10^6\ \text{mm}^4$ and $I_{zy} = -0.74 \times 10^6\ \text{mm}^4$. Determine the magnitude and direction of the deflection at the mid-span section of the beam. Take $E = 70\ 000\ \text{N/mm}^2$.

Ans. 52.3 mm at 23.9° below horizontal.

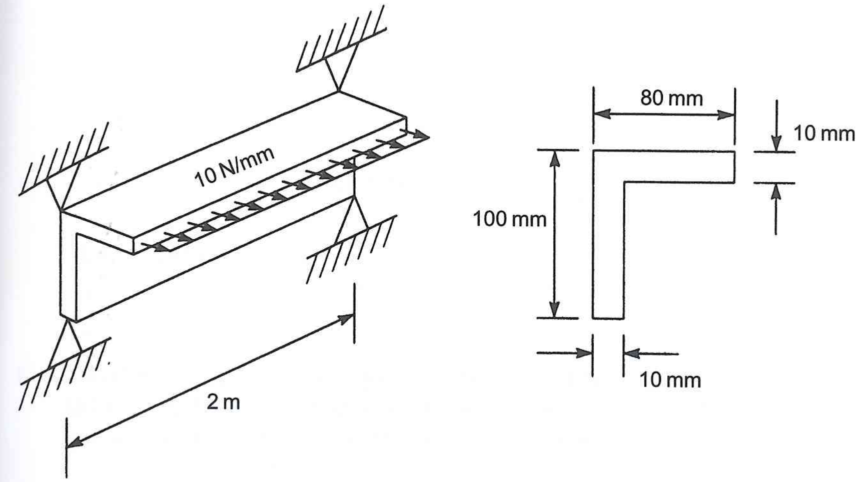


FIGURE P.13.13

P.13.14 A uniform cantilever of arbitrary cross section and length L has section properties I_z, I_y and I_{zy} with respect to the centroidal axes shown (Fig. P.13.14). It is loaded in the vertical plane by a tip load W. The tip of the beam is hinged to a horizontal link which constrains it to move in the vertical direction only (provided that the actual deflections are small). Assuming that the link is rigid and that there are no twisting effects, calculate the force in the link and the deflection of the tip of the beam.

Ans. WI_{zy}/I_z (compression if I_{zy} is positive), $WL^3/3EI_z$ (downwards).

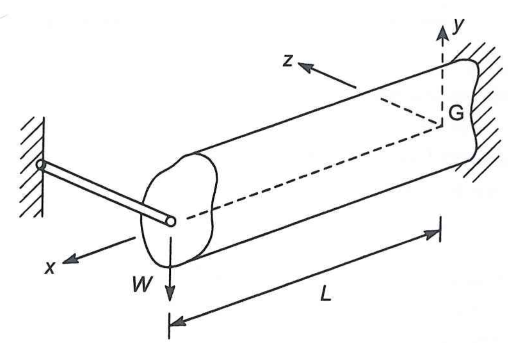


FIGURE P.13.14

P.13.15 A thin-walled beam has the cross section shown in Fig. P.13.15 and is simply supported over a span of 2 m. If the beam carries a horizontal uniformly distributed load of 10 kN/m applied in

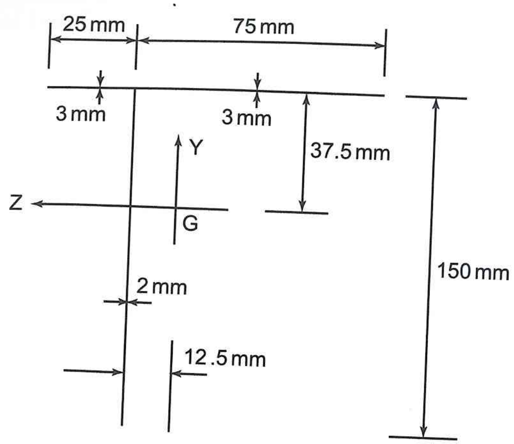


FIGURE P.13.15

the plane of its leg, both loads being over the complete span, calculate the magnitude and direction of the deflection of the mid-span point. Take $E = 200000$ N/mm^2.

Ans. 24.4 mm at 64.8° to the right of vertical.

P.13.16 A thin-walled beam is simply supported at each end and supports a uniformly distributed load of intensity w per unit length in the plane of its lower horizontal flange (see Fig. P.13.16). Calculate the horizontal and vertical components of the deflection of the mid-span point. Take $E = 200\ 000$ N/mm^2.

Ans. $u = -9.1$ mm, $v = 5.2$ mm.

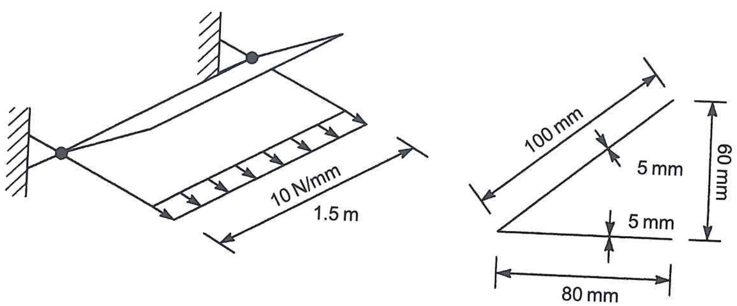


FIGURE P.13.16

P.13.17 A uniform beam of arbitrary unsymmetrical cross section and length $2L$ is built-in at one end and is simply supported in the vertical direction at a point half-way along its length. This support, however, allows the beam to deflect freely in the horizontal z direction (Fig. P.13.17). Determine the vertical reaction at the support.

Ans. 5 W/2.

P.13.18 A cantilever of length $3L$ has section second moments of area I_z, I_y and I_{zy} referred to horizontal and vertical axes through the centroid of its cross section. If the cantilever carries a vertically downward load W at its free end and is pinned to a support which prevents both vertical and horizontal movement at a distance $2L$ from the built-in end, calculate the magnitude of the vertical reaction at the support. Show also that the horizontal reaction is zero.

Ans. $7W/4$.

P.13.19 A b...

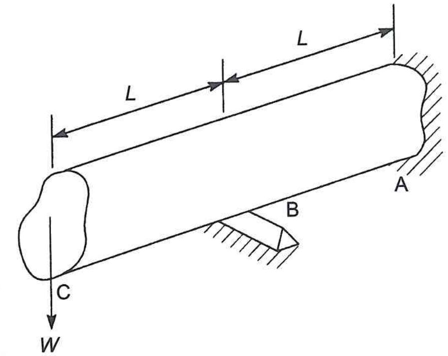


FIGURE P.13.17

downward load of 200 kN at the free end of the overhang. Calculate the deflection of the beam midway between the supports allowing for the effects of both bending and shear. Take $E = 200000$ N/mm^2 and $G = 70000$ N/mm^2. What percentage of the total deflection is due to shear?

Ans. 1.03 mm upwards, 8.7%.

P.13.20 Calculate the deflection due to shear at the mid-span point of a simply supported rectangular section beam of length L which carries a vertically downward load W at mid-span. The beam has a cross section of breadth B and depth D; the shear modulus is G.

Ans. $3WL/10GBD$ (downwards).

P.13.21 Determine the deflection due to shear at the free end of a cantilever of length L and rectangular cross section $B \times D$ which supports a uniformly distributed load of intensity w. The shear modulus is G.

Ans. $3wL^2/5GBD$ (downwards).

P.13.22 A cantilever of length L has a solid circular cross section of diameter D and carries a vertically downward load W at its free end. The modulus of rigidity of the cantilever is G. Calculate the shear stress distribution across a section of the cantilever and hence determine the deflection due to shear at its free end.

Ans. $\tau = 16W(1-4y^2/D^2)/3\pi D^2$, $40WL/9\pi GD^2$ (downwards).

P.13.23 Show that the deflection due to shear in a rectangular section beam supporting a vertical shear load S_y is 20% greater for a shear stress distribution given by the expression

$$\tau = -\frac{S_y A'\bar{y}}{b_o I_z}$$

than for a distribution assumed to be uniform.

A rectangular section cantilever beam 200 mm wide by 400 mm deep and 2 m long carries a vertically downward load of 500 kN at a distance of 1 m from its free end. Calculate the deflection at the free end taking into account both shear and bending effects. Take $E = 200\ 000$ N/mm^2 and $G = 70\ 000$ N/mm^2.

Ans. 2.06 mm (downwards).

P.13.24 Determine the form factor β for the beam section shown in Fig. P.13.24.

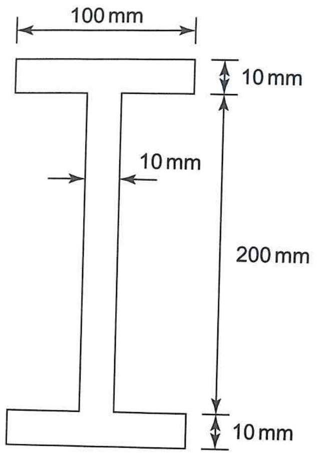


FIGURE P.13.24

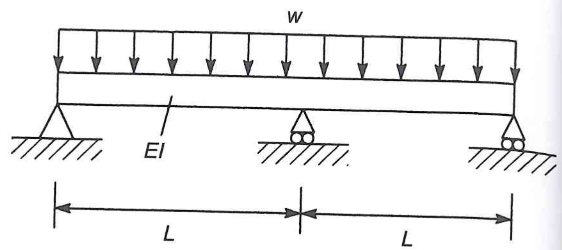


FIGURE P.13.26

P.13.25 A cantilever beam of length L has a solid circular cross section of diameter D and carries a vertically downward load W at its free end. Calculate the distribution of shear stress in a cross section of the beam and hence the form factor β. What is the deflection due to shear at the free end of the cantilever? The shear modulus is G and note that

$\int_{-\pi/2}^{\pi/2} \cos^6 \theta \, d\theta = 5\pi/16$

Ans. $\beta = 10/9$, $40WL/9\pi GD^2$.

P.13.26 The beam shown in Fig. P.13.26 is simply supported at each end and is provided with an additional support at mid-span. If the beam carries a uniformly distributed load of intensity w and has a flexural rigidity EI, use the principle of superposition to determine the reactions in the supports.

Ans. $5wL/4$ (central support), $3wL/8$ (outside supports).

P.13.27 A built-in beam ACB of span L carries a concentrated load W at C a distance a from A and b from B. If the flexural rigidity of the beam is EI, use the principle of superposition to determine the support reactions.

Ans. $R_A = Wb^2(L + 2a)/L^3$, $R_B = Wa^2(L + 2b)/L^3$, $M_A = Wab^2/L^2$, $M_B = Wa^2b/L^2$.

P.13.28 A beam has a second moment of area I for the central half of its span and $I/2$ for the outer quarters. If the beam carries a central concentrated load W, find the deflection at mid-span if the beam is simply supported and also the fixed-end moments when both ends of the beam are built-in.

Ans. $3WL^3/128EI$, $5WL/48$.

P.13.29 A cantilever beam projects 1.5 m from its support and carries a uniformly distributed load of 16 kN/m over its whole length together with a load of 30 kN at 0.75 m from the support. The outer end rests on a prop which compresses 0.12 mm for every kN of compressive load. If the value of EI for the beam is 2000 kNm^2, determine the reaction in the prop.

Ans. 23.4 kN.

CHAPTER 14

Complex Stress and Strain

In Chapters 7, 9, 10 and 11 we determined stress distributions produced separately by axial load, bending moment, shear force and torsion. However, in many practical situations some or all of these force systems act simultaneously so that the various stresses are combined to form complex systems which may include both direct and shear stresses. In such cases it is no longer a simple matter to predict the mode of failure of a structural member, particularly since, as we shall see, the direct and shear stresses at a point due to, say, bending and torsion combined are not necessarily the maximum values of direct and shear stress at that point.

Therefore as a preliminary to the investigation of the theories of elastic failure in Section 14.10 we shall examine states of stress and strain at points in structural members subjected to complex loading systems.

14.1 Representation of stress at a point

We have seen that, generally, stress distributions in structural members vary throughout the member. For example the direct stress in a cantilever beam carrying a point load at its free end varies along the length of the beam and throughout its depth. Suppose that we are interested in the state of stress at a point lying in the vertical plane of symmetry and on the upper surface of the beam mid-way along its span. The direct stress at this point on planes perpendicular to the axis of the beam can be calculated using Eq. (9.9). This stress may be imagined to be acting on two opposite sides of a very small thin element ABCD in the surface of the beam at the point (Fig. 14.1).

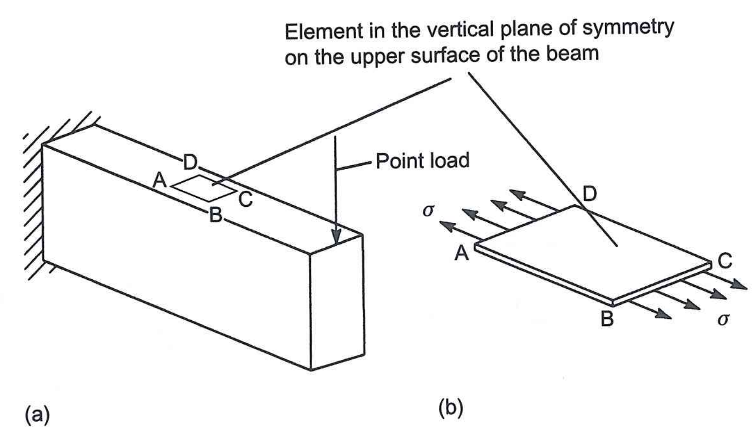


FIGURE 14.1

Representation of stress at a point in a structural member.

Since the element is thin we can ignore any variation in direct stress across its thickness. Similarly, since the sides of the element are extremely small we can assume that σ has the same value on each opposite side BC and AD of the element and that σ is constant along these sides (in this particular case σ is constant across the width of the beam but the argument would apply if it were not). We are therefore representing the stress at a point in a structural member by a stress system acting on the sides and in the plane of a thin, very small element; such an element is known as a two-dimensional element and the stress system is a plane stress system as we saw in Section 7.11.

14.2 Determination of stresses on inclined planes

Suppose that we wish to determine the direct and shear stresses at the same point in the cantilever beam of Fig. 14.1 but on a plane PQ inclined at an angle to the axis of the beam as shown in Fig. 14.2(a). The direct stress on the sides AD and BC of the element ABCD is σ_x in accordance with the sign convention adopted previously.

Consider the triangular portion PQR of the element ABCD where QR is parallel to the sides AD and BC. On QR there is a direct stress which must also be σ_x since there is no variation of direct stress on planes parallel to QR between the opposite sides of the element. On the side PQ of the triangular element let σ_n be the direct stress and τ the shear stress. Although the stresses are uniformly distributed along the sides of the elements it is convenient to represent them by single arrows as shown in Fig. 14.2(b).

The triangular element PQR is in equilibrium under the action of forces corresponding to the stresses σ_x, σ_n and τ. Thus, resolving forces in a direction perpendicular to PQ and assuming that the element is of unit thickness we have

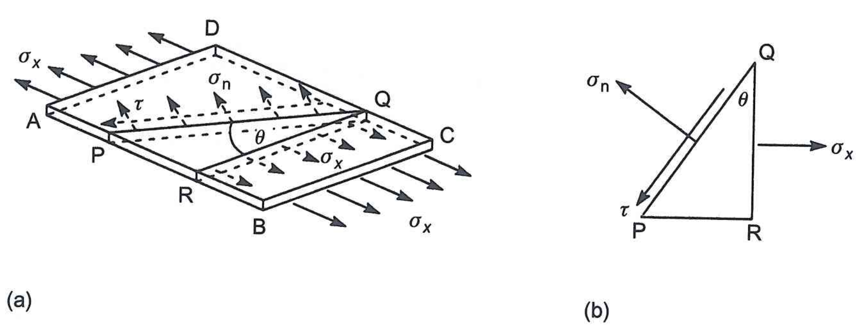
$$\sigma_n \, PQ = \sigma_x \, QR \cos\theta$$

or

$$\sigma_n = \sigma_x \frac{QR}{PQ} \cos\theta$$

which simplifies to

$$\sigma_n = \sigma_x \cos^2\theta \qquad (14.1)$$



(a) (b)

FIGURE 14.2

Determination of stresses on an inclined plane

Resolving forces parallel to PQ

$$\tau \, PQ = \sigma_x QR \sin\theta$$

from which

$$\tau = \sigma_x \cos\theta \sin\theta$$

or

$$\tau = \frac{\sigma_x}{2} \sin 2\theta \qquad (14.2)$$

We see from Eqs (14.1) and (14.2) that although the applied load induces direct stresses only on planes perpendicular to the axis of the beam, both direct and shear stresses exist on planes inclined to the axis of the beam. Furthermore it can be seen from Eq. (14.2) that the shear stress τ is a maximum when $\theta = 45°$. This explains the mode of failure of ductile materials subjected to simple tension and other materials such as timber under compression. For example, a flat aluminium alloy test piece fails in simple tension along a line at approximately 45° to the axis of loading as illustrated in Fig. 14.3. This suggests that the crystal structure of the metal is relatively weak in shear and that failure takes the form of sliding of one crystal plane over another as opposed to the tearing apart of two crystal planes. The failure is therefore a shear failure although the test piece is in simple tension.

Biaxial stress system

A more complex stress system may be produced by a loading system such as that shown in Fig. 14.4 where a thin-walled hollow cylinder is subjected to an internal pressure, p. The internal pressure induces circumferential or hoop stresses σ_y, given by Eq. (7.63), on planes parallel to the axis of the cylinder and, in addition, longitudinal stresses, σ_x, on planes perpendicular to the axis of the cylinder (Eq. (7.62)). Thus any two-dimensional element of unit thickness in the wall of the cylinder and having sides perpendicular and parallel to the axis of the cylinder supports a biaxial stress system as shown in Fig. 14.4. In this particular case σ_x and σ_y each have constant values irrespective of the position of the element.

Let us consider the equilibrium of a triangular portion ABC of the element as shown in Fig. 14.5. Resolving forces in a direction perpendicular to AB we have

$$\sigma_n AB = \sigma_x BC \cos\theta + \sigma_y AC \sin\theta$$

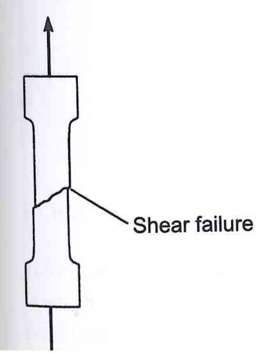


FIGURE 14.3

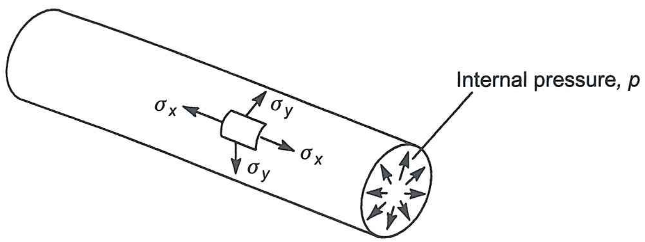


FIGURE 14.4

Generation of a biaxial stress system.

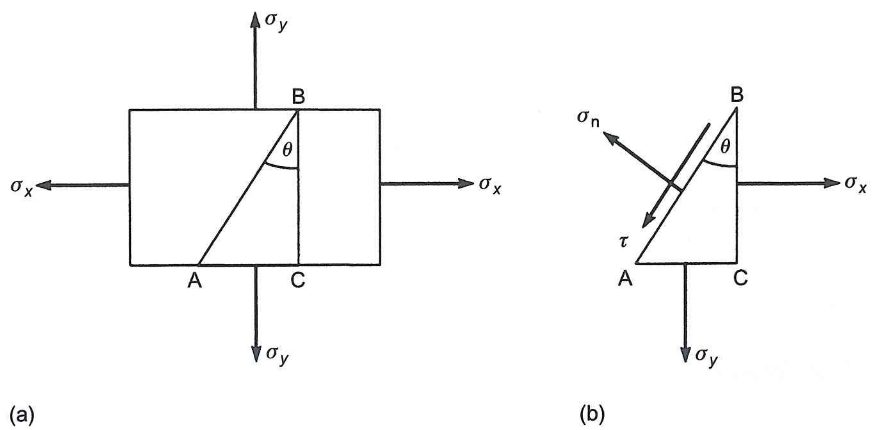


FIGURE 14.5

Determination of stresses on an inclined plane in a biaxial stress system.

or

$$\sigma_n = \sigma_x \frac{BC}{AB} \cos\theta + \sigma_y \frac{AC}{AB} \sin\theta$$

which gives

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta \tag{14.3}$$

Resolving forces parallel to AB

$$\tau\, AB = \sigma_x BC \sin\theta - \sigma_y AC \cos\theta$$

or

$$\tau = \sigma_x \frac{BC}{AB} \sin\theta - \sigma_y \frac{AC}{AB} \cos\theta$$

which gives

$$\tau = \left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta \tag{14.4}$$

Again we see that although the applied loads produce only direct stresses on planes perpendicular and parallel to the axis of the cylinder, both direct and shear stresses exist on inclined planes. Furthermore, for given values of σ_x and σ_y (i.e. p) the shear stress τ is a maximum on planes inclined

EXAMPLE 14.1

A cylindrical pressure vessel has an internal diameter of 2 m and is fabricated from plates 20 mm thick. If the pressure inside the vessel is 1.5 N/mm^2 and, in addition, the vessel is subjected to an axial tensile load of 2500 kN, calculate the direct and shear stresses on a plane inclined at an angle of 60° to the axis of the vessel. Calculate also the maximum shear stress.

From Eq. (7.63) the circumferential stress is

$$\frac{pd}{2t} = \frac{1.5 \times 2 \times 10^3}{2 \times 20} = 75\ \text{N/mm}^2$$

From Eq. (7.62) the longitudinal stress is

$$\frac{pd}{4t} = 37.5\ \text{N/mm}^2$$

The direct stress due to axial load is, from Eq. (7.1)

$$\frac{2500 \times 10^3}{\pi \times 2000 \times 20} = 19.9\ \text{N/mm}^2$$

Therefore on a rectangular element at any point in the wall of the vessel there is a biaxial stress system as shown in Fig. 14.6. Now considering the equilibrium of the triangular element ABC we have, resolving forces perpendicular to AB

$$\sigma_n\, AB \times 20 = 57.4\, BC \times 20 \cos 30^\circ + 75 AC \times 20 \cos 60^\circ$$

Since the walls of the vessel are thin the thickness of the two-dimensional element may be taken as 20 mm. However, as can be seen, the thickness cancels out of the above equation so that it is simpler to assume unit thickness for two-dimensional elements in all cases. Then

$$\sigma_n = 57.4 \cos^2 30^\circ + 75 \cos^2 60^\circ$$

which gives

$$\sigma_n = 61.8\ \text{N/mm}^2$$

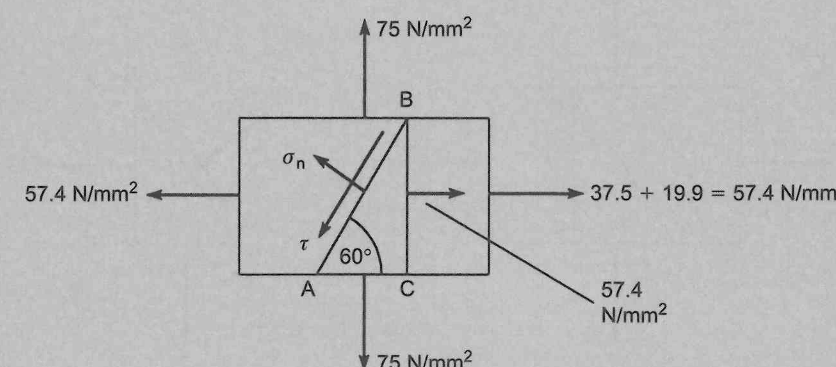


FIGURE 14.6

Biaxial stress system of Ex. 14.1.

Resolving parallel to AB

$$\tau\,AB = 57.4\,BC\cos 60° - 75\,AC\sin 60°$$

or

$$\tau = 57.4\sin 60°\cos 60° - 75\cos 60°\sin 60°$$

from which

$$\tau = -7.6\ \text{N/mm}^2$$

The negative sign of τ indicates that τ acts in the direction AB and not, as was assumed, in the direction BA. From Eq. (14.4) it can be seen that the maximum shear stress occurs on planes inclined at 45° to the axis of the cylinder and is given by

$$\tau_{max} = \frac{57.4 - 75}{2} = -8.8\ \text{N/mm}^2$$

Again the negative sign of τ_{max} indicates that the direction of τ_{max} is opposite to that assumed.

General two-dimensional case

If we now apply a torque to the cylinder of Fig. 14.4 in a clockwise sense when viewed from the right-hand end, shear and complementary shear stresses are induced on the sides of the rectangular element in addition to the direct stresses already present. The value of these shear stresses is given by Eq. (11.21) since the cylinder is thin-walled. We now have a general two-dimensional stress system acting on the element as shown in Fig. 14.7(a). The suffixes employed in designating shear stress refer to the plane on which the stress acts and its direction. Thus τ_{xy} is a shear stress acting on an x plane in the y direction. Conversely τ_{yx} acts on a y plane in the x direction. However, since $\tau_{xy} = \tau_{yx}$ we label both shear and complementary shear stresses τ_{xy} as in Fig. 14.7(b).

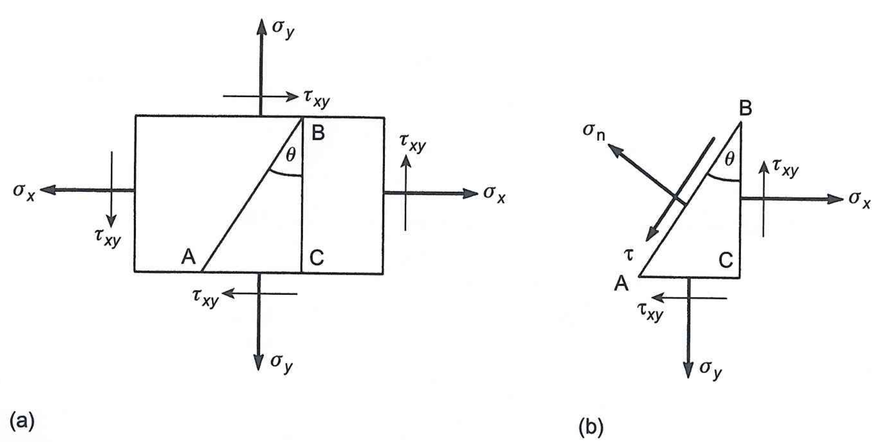


FIGURE 14.7

Considering the equilibrium of the triangular element ABC in Fig. 14.7(b) and resolving forces in a direction perpendicular to AB

$$\sigma_n\,AB = \sigma_x\,BC\cos\theta + \sigma_y\,AC\sin\theta + \tau_{xy}\,BC\sin\theta - \tau_{xy}\,AC\cos\theta$$

Dividing through by AB and simplifying we obtain

$$\sigma_n = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta - \tau_{xy}\sin 2\theta \tag{14.5}$$

Now resolving forces parallel to BA

$$\tau\,AB = \sigma_x\,BC\sin\theta - \sigma_y\,AC\cos\theta + \tau_{xy}\,BC\cos\theta - \tau_{xy}\,AC\sin\theta$$

Again dividing through by AB and simplifying we have

$$\tau = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \tag{14.6}$$

EXAMPLE 14.2

A cantilever of solid, circular cross section supports a compressive load of 50 000 N applied to its free end at a point 1.5 mm below a horizontal diameter in the vertical plane of symmetry together with a torque of 1200 Nm (Fig. 14.8).

Calculate the direct and shear stresses on a plane inclined at 60° to the axis of the cantilever at a point on the lower edge of the vertical plane of symmetry.

The direct loading system is equivalent to an axial load of 50 000 N together with a bending moment of 50 000 × 1.5 = 75 000 N mm in a vertical plane. Thus at any point on the lower edge of the vertical plane of symmetry there are direct compressive stresses due to axial load and bending moment which act on planes perpendicular to the axis of the beam and are given, respectively, by Eqs (7.1) and (9.9). Therefore

$$\sigma_x(\text{axial load}) = \frac{50\,000}{\pi \times 60^2/4} = 17.7\ \text{N/mm}^2$$

$$\sigma_x(\text{bending moment}) = \frac{75\,000 \times 30}{\pi \times 60^4/64} = 3.5\ \text{N/mm}^2$$

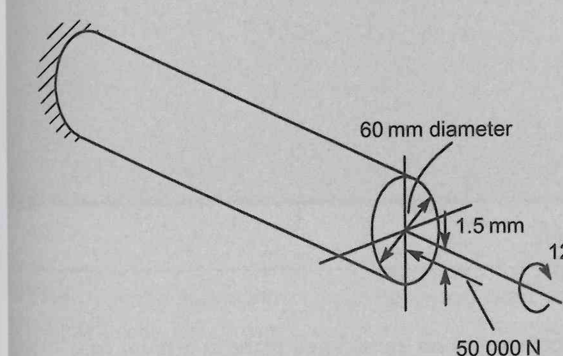


FIGURE 14.8

Cantilever beam of Ex. 14.2.

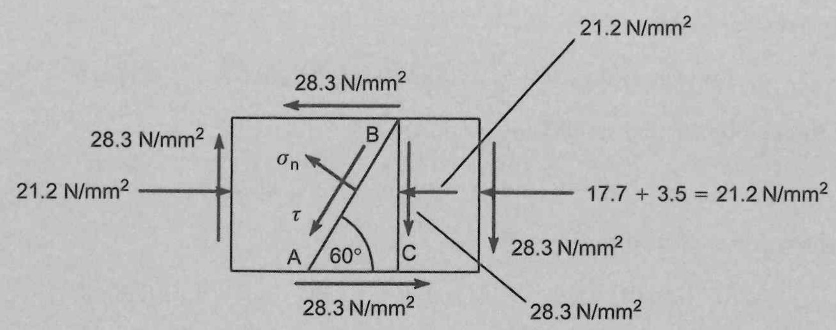


FIGURE 14.9

Two-dimensional stress system in cantilever beam of Ex. 14.2.

The shear stress τ_{xy} at the same point due to the torque is obtained from Eq. (11.4) and is

$$\tau_{xy} = \frac{1200 \times 10^3 \times 30}{\pi \times 60^4/32} = 28.3\ \text{N/mm}^2$$

The stress system acting on a two-dimensional rectangular element at the point is as shown in Fig. 14.9. Note that, in this case, the element is at the bottom of the cylinder so that the shear stress is opposite in direction to that in Fig. 14.7. Considering the equilibrium of the triangular element ABC and resolving forces in a direction perpendicular to AB we have

$$\sigma_n\,\text{AB} = -\,21.2\ \text{BC} \cos 30° + 28.3\ \text{BC} \sin 30° + 28.3\ \text{AC} \cos 30°$$

Dividing through by AB we obtain

$$\sigma_n = -21.2 \cos^2 30° + 28.3 \cos 30° \sin 30° + 28.3 \sin 30° \cos 30°$$

which gives

$$\sigma_n = 8.6\ \text{N/mm}^2$$

Similarly resolving parallel to AB

$$\tau\,\text{AB} = -21.2\ \text{BC} \cos 60° - 28.3\ \text{BC} \sin 60° + 28.3\ \text{AC} \cos 60°$$

so that

$$\tau = -21.2 \sin 60° \cos 60° - 28.3 \sin^2 60° + 28.3 \cos^2 60°$$

from which

$$\tau = -23.3\ \text{N/mm}^2$$

acting in the direction AB.

14.3 Principal stresses

Equations (14.5) and (14.6) give the direct and shear stresses on an inclined plane at a point in a struc-

system at that point. Clearly for given values of σ_x, σ_y and τ_{xy}, in other words a given loading system, both σ_n and τ vary with the angle θ and will attain maximum or minimum values when $d\sigma_n/d\theta = 0$ and $d\tau/d\theta = 0$. From Eq. (14.5)

$$\frac{d\sigma_n}{d\theta} = -2\sigma_x \cos\theta \sin\theta + 2\sigma_y \sin\theta \cos\theta - 2\tau_{xy} \cos 2\theta = 0$$

then

$$-(\sigma_x - \sigma_y) \sin 2\theta - 2\tau_{xy} \cos 2\theta = 0$$

or

$$\tan 2\theta = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{14.7}$$

Two solutions, $-\theta$ and $-\theta - \pi/2$, satisfy Eq. (14.7) so that there are two mutually perpendicular planes on which the direct stress is either a maximum or a minimum. Furthermore, by comparison of Eqs (14.7) and (14.6) it can be seen that these planes correspond to those on which $\tau = 0$.

The direct stresses on these planes are called *principal stresses* and the planes are called *principal planes*.

From Eq. (14.7)

$$\sin 2\theta = -\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \qquad \cos 2\theta = \frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

and

$$\sin 2\left(\theta + \frac{\pi}{2}\right) = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\cos 2\left(\theta + \frac{\pi}{2}\right) = \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Rewriting Eq. (14.5) as

$$\sigma_n = \frac{\sigma_x}{2}(1 + \cos 2\theta) + \frac{\sigma_y}{2}(1 - \cos 2\theta) - \tau_{xy} \sin 2\theta$$

and substituting for {sin 2θ, cos 2θ} and {sin $2(\theta + \pi/2)$, cos $2(\theta + \pi/2)$} in turn gives

$$\sigma_{\text{I}} = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \tag{14.8}$$

$$\sigma_{\text{II}} = \frac{\sigma_x + \sigma_y}{2} - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \tag{14.9}$$

where σ_{I} is the *maximum* or *major principal stress* and σ_{II} is the *minimum* or *minor principal stress*; σ_{I} is algebraically the greatest direct stress at the point while σ_{II} is algebraically the least. Note that when σ_{II} is compressive, i.e. negative, it is possible for σ_{II} to be numerically greater than σ_{I}.

From Eq. (14.6)

$$\frac{d\tau}{d\theta} = (\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

giving

$$\tan 2\theta = \frac{(\sigma_x - \sigma_y)}{2\tau_{xy}} \qquad (14.10)$$

It follows that

$$\sin 2\theta = \frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\cos 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\sin 2\left(\theta + \frac{\pi}{2}\right) = -\frac{(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

$$\cos 2\left(\theta + \frac{\pi}{2}\right) = -\frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Substituting these values in Eq. (14.6) gives

$$\tau_{\max,\min} = \pm\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \qquad (14.11)$$

Here, as in the case of the principal stresses, we take the maximum value as being the greater value algebraically.

Comparing Eq. (14.11) with Eqs (14.8) and (14.9) we see that

$$\tau_{\max} = \frac{\sigma_{\mathrm{I}} - \sigma_{\mathrm{II}}}{2} \qquad (14.12)$$

Equations (14.11) and (14.12) give alternative expressions for the maximum shear stress acting at the point *in the plane of the given stresses*. This is not necessarily the maximum shear stress in a three-dimensional element subjected to a two-dimensional stress system, as we shall see in Section 14.10.

Since Eq. (14.10) is the negative reciprocal of Eq. (14.7), the angles given by these two equations differ by 90° so that the planes of maximum shear stress are inclined at 45° to the principal planes.

We see now that the direct stresses, σ_x, σ_y, and shear stresses, τ_{xy}, are not, in a general case, the greatest values of direct and shear stress at the point. This fact is clearly important in designing structural members subjected to complex loading systems, as we shall see in Section 14.10. We can illustrate the stresses acting on the various planes at the point by considering a series of elements at the point as shown in Fig. 14.10. Note that generally there will be a direct stress on the planes on which $\tau_{\max}$ acts.

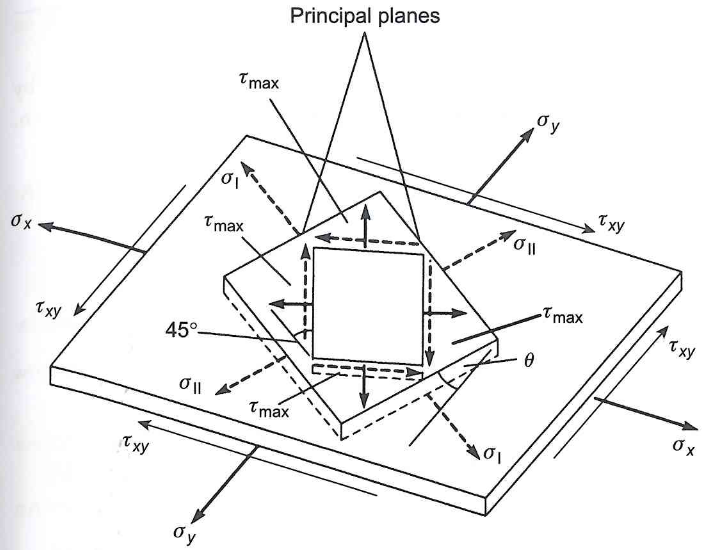


FIGURE 14.10

Stresses acting on different planes at a point in a structural member.

EXAMPLE 14.3

A structural member supports loads which produce, at a particular point, a direct tensile stress of 80 N/mm² and a shear stress of 45 N/mm² on the same plane. Calculate the values and directions of the principal stresses at the point and also the maximum shear stress, stating on which planes this will act.

Suppose that the tensile stress of 80 N/mm² acts in the x direction. Then $\sigma_x = +80$ N/mm², $\sigma_y = 0$ and $\tau_{xy} = 45$ N/mm². Substituting these values in Eqs (14.8) and (14.9) in turn gives

$$\sigma_{\mathrm{I}} = \frac{80}{2} + \frac{1}{2}\sqrt{80^2 + 4\times 45^2} = 100.2\ \mathrm{N/mm^2}$$

$$\sigma_{\mathrm{II}} = \frac{80}{2} - \frac{1}{2}\sqrt{80^2 + 4\times 45^2} = -20.2\ \mathrm{N/mm^2}$$

From Eq. (14.7)

$$\tan 2\theta = -\frac{2\times 45}{80} = -1.125$$

from which

$$\theta = -24°11' \quad (\text{corresponding to } \sigma_{\mathrm{I}})$$

Also, the plane on which σ_{II} acts corresponds to $\theta = -24°11' - 90° = -114°11'$.

The maximum shear stress is most easily found from Eq. (14.12) and is given by

$$\tau_{\max} = \frac{100.2 - (-20.2)}{2} = 60.2\ \mathrm{N/mm^2}$$

The maximum shear stress acts on planes at 45° to the principal planes. Thus $\theta = -69°11'$ and $\theta = -159°11'$ give the planes of maximum shear stress.

14.4 Mohr's circle of stress

The state of stress at a point in a structural member may be conveniently represented graphically by *Mohr's circle of stress*. We have shown that the direct and shear stresses on an inclined plane are given, in terms of known applied stresses, by

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta - \tau_{xy}\sin 2\theta \text{ (Eq. (14.5))}$$

and

$$\tau = \frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \text{ (Eq. (14.6))}$$

respectively. The positive directions of these stresses and the angle θ are shown in Fig. 14.7. We now write Eq. (14.5) in the form

$$\sigma_n = \frac{\sigma_x}{2}(1 + \cos 2\theta) + \frac{\sigma_y}{2}(1 - \cos 2\theta) - \tau_{xy}\sin 2\theta$$

or

$$\sigma_n = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy}\sin 2\theta \tag{14.13}$$

Now squaring and adding Eqs (14.6) and (14.13) we obtain

$$\left[\sigma_n - \frac{1}{2}(\sigma_x + \sigma_y)\right]^2 + \tau^2 = \left[\frac{1}{2}(\sigma_x - \sigma_y)\right]^2 + \tau_{xy}^2 \tag{14.14}$$

Equation (14.14) represents the equation of a circle of radius

$$\pm\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

and having its centre at the point $\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$.

The circle may be constructed by locating the points $Q_1(\sigma_x, -\tau_{xy})$ and $Q_2(\sigma_y, +\tau_{xy})$ referred to axes $O\sigma\tau$ as shown in Fig. 14.11. The line Q_1Q_2 is then drawn and intersects the $O\sigma$ axis at C. From Fig. 14.11

$$OC = OP_1 - CP_1 = \sigma_x - \frac{\sigma_x - \sigma_y}{2}$$

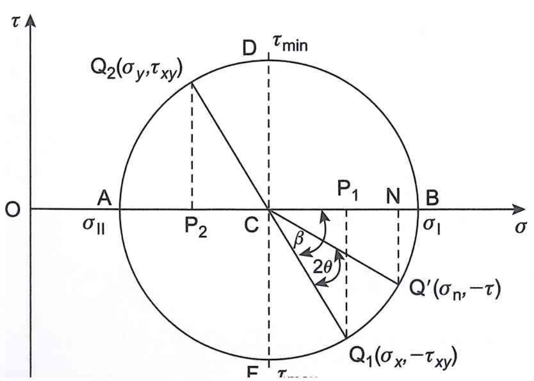


FIGURE 14.11

so that

$$OC = \frac{\sigma_x + \sigma_y}{2}$$

Thus the point C has coordinates $\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$ which, as we have seen, is the centre of the circle.

Also

$$CQ_1 = \sqrt{CP_1^2 + P_1Q_1^2} = \sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2}$$

whence

$$CQ_1 = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

which is the radius of the circle; the circle is then drawn as shown.

Now we set CQ′ at an angle 2θ (positive clockwise) to CQ_1; Q′ is then the point $(\sigma_n, -\tau)$ as demonstrated below.

From Fig. 14.11 we see that

$$ON = OC + CN$$

or, since

$$OC = (\sigma_x + \sigma_y)/2, \quad CN = CQ'\cos(\beta - 2\theta) \text{ and } CQ' = CQ_1,$$

we have

$$\sigma_n = \frac{\sigma_x - \sigma_y}{2} + CQ_1(\cos\beta\cos 2\theta + \sin\beta\sin 2\theta)$$

But

$$CQ_1 = \frac{CP_1}{\cos\beta} \quad \text{and} \quad CP_1 = \frac{\sigma_x - \sigma_y}{2}$$

Hence

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + CP_1\tan\beta\sin 2\theta$$

which, on rearranging, becomes

$$\sigma_n = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta - \tau_{xy}\sin 2\theta$$

as in Eq. (14.5). Similarly it may be shown that

$$Q'N = -\tau_{xy}\cos 2\theta - \left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta = -\tau$$

as in Eq. (14.6). It must be remembered that the construction of Fig. 14.11 corresponds to the stress system of Fig. 14.7(b); any sign reversal must be allowed for. Also the $O\sigma$ and $O\tau$ axes must be constructed to the same scale otherwise the circle would not be that represented by Eq (14.14).

The maximum and minimum values of the direct stress σ_n, that is the major and minor principal stresses σ_I and σ_{II}, occur when N and Q′ coincide with B and A, respectively. Thus i.e.

$$\sigma_I = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad \text{(as in Eq. (14.8))}$$

and

$$\sigma_{II} = \text{OC} - \text{radius of the circle}$$

so that

$$\sigma_{II} = \frac{\sigma_x + \sigma_y}{2} - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad \text{(as in Eq. (14.9))}$$

The principal planes are then given by $2\theta = \beta(\sigma_I)$ and $2\theta = \beta + \pi\ (\sigma_{II})$.

The maximum and minimum values of the shear stress τ occur when Q' coincides with F and D at the lower and upper extremities of the circle. At these points $\tau_{max,min}$ are clearly equal to the radius of the circle. Hence

$$\tau_{max,min} = \pm\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \quad \text{(see Eq. (14.11))}$$

The minimum value of shear stress is the algebraic minimum. The planes of maximum and minimum shear stress are given by $2\theta = \beta + \pi/2$ and $2\theta = \beta + 3\pi/2$ and are inclined at 45° to the principal planes.

EXAMPLE 14.4

Direct stresses of 160 N/mm^2, tension, and 120 N/mm^2, compression, are applied at a particular point in an elastic material on two mutually perpendicular planes. The maximum principal stress in the material is limited to 200 N/mm^2, tension. Use a graphical method to find the allowable value of shear stress at the point.

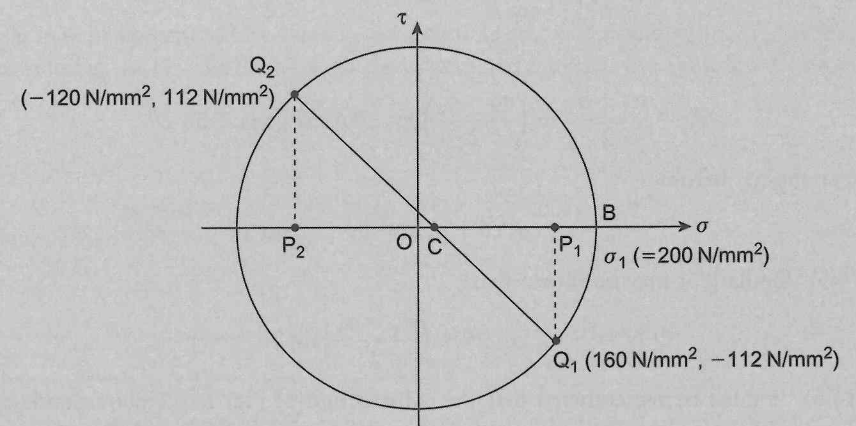
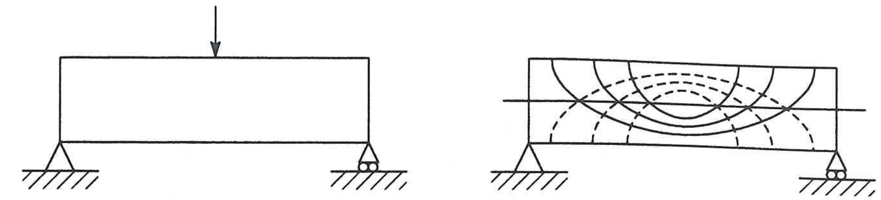


FIGURE 14.12

Mohr's circle of stress for Ex. 14.4.



(a) (b)

FIGURE 14.13

Stress trajectories in a beam.

First, axes Oσ τ are set up to a suitable scale. P_1 and P_2 are then located corresponding to σ_x = 160 N/mm^2 and σ_y = −120 N/mm^2, respectively; the centre C of the circle is mid-way between P_1 and P_2 (Fig. 14.12). The radius is obtained by locating B(σ_1 = 200 N/mm^2) and the circle then drawn. The maximum allowable applied shear stress, τ_{xy}, is then obtained by locating Q_1 or Q_2. The maximum shear stress at the point is equal to the radius of the circle and is 180 N/mm^2.

14.5 Stress trajectories

We have shown that direct and shear stresses at a point in a beam produced, say, by bending and shear and calculated by the methods discussed in Chapters 9 and 10, respectively, are not necessarily the greatest values of direct and shear stress at the point. In order, therefore, to obtain a more complete picture of the distribution, magnitude and direction of the stresses in a beam we investigate the manner in which the principal stresses vary throughout a beam.

Consider the simply supported beam of rectangular section carrying a central concentrated load as shown in Fig. 14.13(a). Using Eqs (9.9) and (10.4) we can determine the direct and shear stresses at any point in any section of the beam. Subsequently from Eqs (14.8), (14.9) and (14.7) we can find the principal stresses at the point and their directions. If this procedure is followed for very many points throughout the beam, curves, to which the principal stresses are tangential, may be drawn as shown in Fig. 14.13(b). These curves are known as *stress trajectories* and form two orthogonal systems; in Fig. 14.13(b) solid lines represent tensile principal stresses and dotted lines compressive principal stresses. The two sets of curves cross each other at right angles and all curves intersect the neutral axis at 45° where the direct stress (calculated from Eq. (9.9)) is zero. At the top and bottom surfaces of the beam where the shear stress (calculated from Eq. (10.4)) is zero the trajectories have either horizontal or vertical tangents.

Another type of curve that may be drawn from a knowledge of the distribution of principal stress is a *stress contour*. Such a curve connects points of equal principal stress.

14.6 Determination of strains on inclined planes

In Section 14.2 we investigated the two-dimensional state of stress at a point in a structural member and determined direct and shear stresses on inclined planes; we shall now determine the accompanying

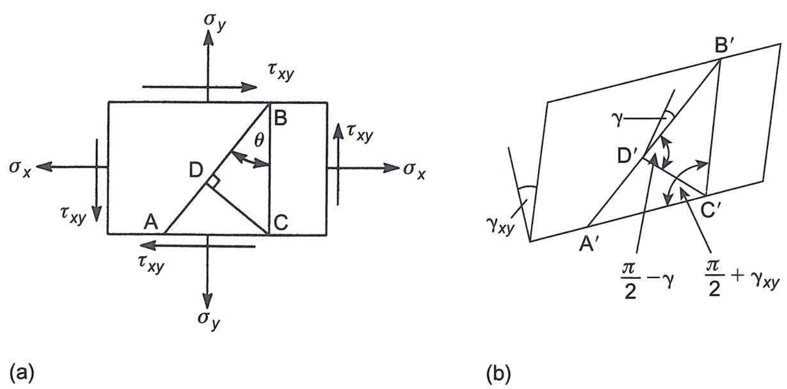


FIGURE 14.14

Determination of strains on an inclined plane.

Figure 14.14(a) shows a two-dimensional element subjected to a complex direct and shear stress system. The applied stresses will distort the rectangular element of Fig. 14.14(a) into the shape shown in Fig. 14.14(b). In particular, the triangular element ABC will suffer distortion to the shape A′B′C with corresponding changes in the length CD and the angle BDC. The strains associated with the stresses σ_x, σ_y and τ_{xy} are ε_x, ε_y and γ_{xy}, respectively. We shall now determine the direct strain ε_n in a direction normal to the plane AB and the shear strain γ produced by the shear stress acting on the plane AB.

To a first order of approximation

$$\left.\begin{aligned} A'C' &= AC(1+\varepsilon_x) \\ C'B' &= CB(1+\varepsilon_y) \\ A'B' &= AC(1+\varepsilon_{n+\pi/2}) \end{aligned}\right\} \qquad (14.15)$$

where $\varepsilon_{n+\pi/2}$ is the direct strain in the direction AB. From the geometry of the triangle A′B′C′ in which angle B′C′A′ $= \pi/2 + \gamma_{xy}$

$$(A'B')^2 = (A'C')^2 + (C'B')^2 - 2(A'C')(C'B')\cos\left(\frac{\pi}{2} + \gamma_{xy}\right)$$

or, substituting from Eq. (14.15)

$$(AB)^2(1+\varepsilon_{n+\pi/2})^2 = (AC)^2(1+\varepsilon_x)^2 + (CB)^2(1+\varepsilon_y)^2 + 2(AC)(CB)(1+\varepsilon_x)(1+\varepsilon_y)\sin\gamma_{xy}$$

Noting that $(AB)^2 = (AC)^2 + (CB)^2$ and neglecting squares and higher powers of small quantities, this equation may be rewritten

$$2(AB)^2\varepsilon_{n+\pi/2} = 2(AC)^2\varepsilon_x + 2(CB)^2\varepsilon_y + 2(AC)(CB)\gamma_{xy}$$

Dividing through by $2(AB)^2$ gives

$$\varepsilon_{n+\pi/2} = \varepsilon_x \sin^2\theta + \varepsilon_y \cos^2\theta + \sin\theta\cos\theta\,\gamma_{xy} \qquad (14.16)$$

The strain ε_n in the direction normal to the plane AB is found by replacing the angle θ in Eq. (14.16) by $\theta - \pi/2$. Hence

$$\varepsilon_n = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta \qquad (14.17)$$

Now from triangle C′D′B′ we have

$$(C'B')^2 = (C'D')^2 + (D'B')^2 - 2(C'D')(D'B')\cos\left(\frac{\pi}{2} - \gamma\right) \qquad (14.18)$$

in which

$$C'B' = CB(1+\varepsilon_y)$$
$$C'D' = CD(1+\varepsilon_n)$$
$$D'B' = DB(1+\varepsilon_{n+\pi/2})$$

Substituting in Eq. (14.18) for C′B′,C′D′ and D′B′ and writing $\cos(\pi/2 - \gamma) = \sin\gamma$ we have

$$\begin{aligned}(CB)^2(1+\varepsilon_y)^2 &= (CD)^2(1+\varepsilon_n)^2 + (DB)^2(1+\varepsilon_{n+\pi/2})^2 \\ &\quad - 2(CD)(DB)(1+\varepsilon_n)(1+\varepsilon_{n+\pi/2})\sin\gamma\end{aligned} \qquad (14.19)$$

Again ignoring squares and higher powers of strains and writing $\sin\gamma = \gamma$, Eq. (14.19) becomes

$$(CB)^2(1+2\varepsilon_y) = (CD)^2(1+2\varepsilon_n) + (DB)^2(1+2\varepsilon_{n+\pi/2}) - 2(CD)(DB)\gamma$$

From Fig. 14.14(a) we see that $(CB)^2 = (CD)^2 + (DB)^2$ and the above equation simplifies to

$$2(CB)^2\varepsilon_y = 2(CD)^2\varepsilon_n + 2(DB)^2\varepsilon_{n+\pi/2} - 2(CD)(DB)\gamma$$

Dividing through by $2(CB)^2$ and rearranging we obtain

$$\gamma = \frac{\varepsilon_n \sin^2\theta + \varepsilon_{n+\pi/2}\cos^2\theta - \varepsilon_y}{\sin\theta\cos\theta}$$

Substitution of ε_n and $\varepsilon_{n+\pi/2}$ from Eqs (14.17) and (14.16) yields

$$\frac{\gamma}{2} = \frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta \qquad (14.20)$$

14.7 Principal strains

From a comparison of Eqs (14.17) and (14.20) with Eqs (14.5) and (14.6) we observe that the former two equations may be obtained from Eqs (14.5) and (14.6) by replacing σ_n by ε_n, σ_x by ε_x, σ_y by ε_y, τ_{xy} by $\gamma_{xy}/2$ and τ by $\gamma/2$. It follows that for each deduction made from Eqs (14.5) and (14.6) concerning σ_n and τ there is a corresponding deduction from Eqs (14.17) and (14.20) regarding ε_n and $\gamma/2$. Thus at a point in a structural member there are two mutually perpendicular planes on which the shear strain γ is zero and normal to which the direct strain is the algebraic maximum or minimum direct strain at the point. These direct strains are the *principal strains* at the point and are given (from a comparison with Eqs (14.8) and (14.9)) by

$$\varepsilon_I = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2} \qquad (14.21)$$

and

$$\varepsilon_{II} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2} \qquad (14.22)$$

Since the shear strain γ is zero on these planes it follows that the shear stress must also be zero and we deduce from Section 14.3 that the directions of the principal strains and principal stresses coincide. The related planes are then determined from Eq. (14.7) or from

$$\tan 2\theta = -\frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} \tag{14.23}$$

In addition the maximum shear strain at the point is given by

$$\left(\frac{\gamma}{2}\right)_{\max} = \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2} \tag{14.24}$$

or

$$\left(\frac{\gamma}{2}\right)_{\max} = \frac{\varepsilon_{\text{I}} - \varepsilon_{\text{II}}}{2} \tag{14.25}$$

(cf. Eqs (14.11) and (14.12)).

EXAMPLE 14.5

At a point in a structural member the stresses on two mutually perpendicular planes are 60 N/mm^2 tension and 30 N/mm^2 compression together with a shear stress of 15 N/mm^2. Calculate the principal stresses at the point, the maximum shear stress and the angle which the plane of maximum principal stress makes with the plane on which the 60 N/mm^2 stress acts. Verify all your answers using a graphical method. If Young's modulus $E = 200000\ \text{N/mm}^2$ and Poisson's ratio $\nu = 0.3$ calculate the principal strains and the maximum shear strain.

We shall designate the 60 N/mm^2 as σ_x. Then, from Eq. (14.8)

$$\sigma_{\text{I}} = \frac{60-30}{2} + \frac{1}{2}\sqrt{[(60+30)^2 + 4\times 15^2]}$$

which gives

$$\sigma_{\text{I}} = 62.4\ \text{N/mm}^2$$

Similarly, from Eq. (14.9)

$$\sigma_{\text{II}} = -32.4\ \text{N/mm}^2$$

Then, from Eq. (14.12)

$$\tau_{\max} = \frac{62.4+32.4}{2} = 47.4\ \text{N/mm}^2$$

Alternatively, $\tau_{\max}$ could have been obtained from Eq. (14.11) using the given values of stress although this would have involved slightly longer computation.

From Eq. (14.7)

$$\tan 2\theta = -\frac{2\times 15}{60+30} = -0.33$$

so that

$$2\theta = -18.4^\circ \quad \text{or} \quad -198.4^\circ$$

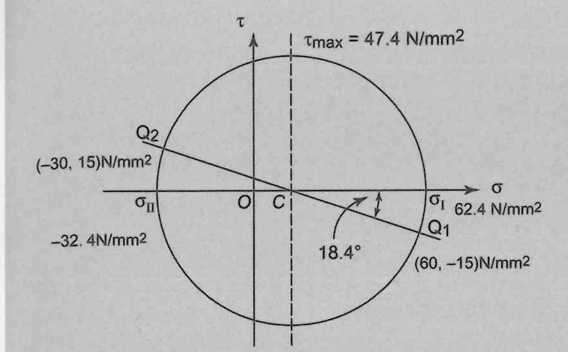


FIGURE 14.15

Mohr's circle for Ex. 14.5.

giving

$$\theta = -9.2^\circ \quad \text{or} \quad -99.2^\circ$$

From Mohr's circle of stress (see Fig. 14.11) it is clear that the plane on which the maximum principal stress acts is at an angle of 9.2° to the plane on which the 60 N/mm^2 stress acts. For this particular problem the solution using Mohr's circle is shown in Fig. 14.15.

From Section 7.8

$$\varepsilon_{\text{I}} = \frac{62.4}{200000} - \frac{0.3(-32.4)}{200000} = 360.6\times 10^{-6}$$

and

$$\varepsilon_{\text{II}} = \frac{-32.4}{200000} - \frac{0.3\times 62.4}{200000} = -255.6\times 10^{-6}$$

and from Eq. (14.25)

$$\frac{(\gamma)_{\max}}{2} = \frac{(360.6+255.6)}{2}\times 10^{-6}$$

which gives

$$\gamma_{\max} = 616.2\times 10^{-6}$$

14.8 Mohr's circle of strain

The argument of Section 14.7 may be applied to Mohr's circle of stress described in Section 14.4. A circle of strain, analogous to that shown in Fig. 14.11, may be drawn when σ_x, σ_y, etc., are replaced by ε_x, ε_y, etc., as specified in Section 14.7. The horizontal extremities of the circle represent the principal strains, the radius of the circle half the maximum shear strain, and so on.

EXAMPLE 14.6

A structural member is loaded in such a way that at a particular point in the member a two-dimensional stress system exists consisting of $\sigma_x = +60\ \text{N/mm}^2$, $\sigma_y = -40\ \text{N/mm}^2$ and $\tau_{xy} = 50\ \text{N/mm}^2$.

a. Calculate the direct strain in the x and y directions and the shear strain, γ_{xy}, at the point.
b. Calculate the principal strains at the point and determine the position of the principal planes.
c. Verify your answer using a graphical method. Take $E = 200\ 000\ \text{N/mm}^2$ and Poisson's ratio, $\nu = 0.3$.

a. From Section 7.8

$$\varepsilon_x = \frac{1}{200\ 000}(60 + 0.3 \times 40) = 360 \times 10^{-6}$$

$$\varepsilon_y = \frac{1}{200\ 000}(-40 - 0.3 \times 60) = -290 \times 10^{-6}$$

The shear modulus, G, is obtained using Eq. (7.21); thus

$$G = \frac{E}{2(1+\nu)} = \frac{200\ 000}{2(1+0.3)} = 76\ 923\ \text{N/mm}^2$$

Hence, from Eq. (7.9)

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{50}{76\ 923} = 650 \times 10^{-6}$$

b. Now substituting in Eqs (14.21) and (14.22) for ε_x, ε_y and γ_{xy} we have

$$\varepsilon_{\mathrm{I}} = 10^{-6}\left[\frac{360 - 290}{2} + \frac{1}{2}\sqrt{(360+290)^2 + 650^2}\right]$$

which gives

$$\varepsilon_{\mathrm{I}} = 495 \times 10^{-6}$$

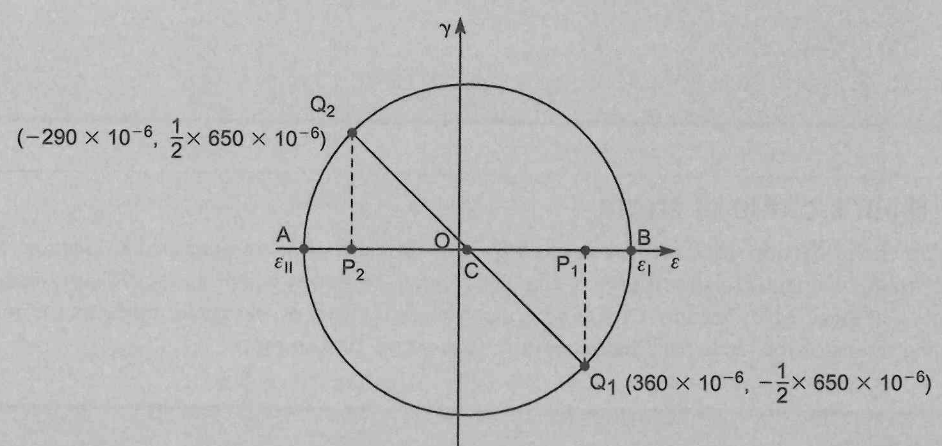


FIGURE 14.16

Mohr's circle of strain for Ex. 14.6.

Similarly

$$\varepsilon_{\mathrm{II}} = -425 \times 10^{-6}$$

From Eq. (14.23) we have

$$\tan 2\theta = -\frac{650 \times 10^{-6}}{360 \times 10^{-6} + 290 \times 10^{-6}} = -1$$

Therefore

$$2\theta = -45^\circ \quad \text{or} \quad -225^\circ$$

so that

$$\theta = -22.5^\circ \quad \text{or} \quad -112.5^\circ$$

c. Axes $O\varepsilon$ and O_γ are set up and the points $Q_1(360 \times 10^{-6}, -\frac{1}{2} \times 650 \times 10^{-6})$ and $Q_2(-290 \times 10^{-6}, \frac{1}{2} \times 650 \times 10^{-6})$ located. The centre C of the circle is the intersection of Q_1Q_2 and the $O\varepsilon$ axis (Fig. 14.16). The circle is then drawn with radius equal to CQ_1 and the points $B(\varepsilon_{\mathrm{I}})$ and $A(\varepsilon_{\mathrm{II}})$ located. Finally, angle $Q_1CB = -2\theta$ and $Q_1CA = -2\theta - \pi$.

14.9 Experimental measurement of surface strains and stresses

Stresses at a point on the surface of a structural member may be determined by measuring the strains at the point, usually with electrical resistance strain gauges. These consist of a short length of fine wire sandwiched between two layers of impregnated paper, the whole being glued to the surface of the member. The resistance of the wire changes as the wire stretches or contracts so that as the surface of the member is strained the gauge indicates a change of resistance which is measurable on a Wheatstone bridge.

Strain gauges measure direct strains only, but the state of stress at a point may be investigated in terms of principal stresses by using a strain gauge 'rosette'. This consists of three strain gauges inclined at a given angle to each other. Typical of these is the 45° or 'rectangular' strain gauge rosette illustrated in Fig. 14.17(a). An equiangular rosette has gauges inclined at 60°.

Suppose that a rosette consists of three arms, 'a', 'b' and 'c' inclined at angles α and β as shown in Fig. 14.17(b). Suppose also that ε_{I} and $\varepsilon_{\mathrm{II}}$ are the principal strains at the point and that ε_{I} is inclined at an unknown angle θ to the arm 'a'. Then if ε_a, ε_b and ε_c are the measured strains in the directions θ, $(\theta + \alpha)$ and $(\theta + \alpha + \beta)$ to ε_{I} we have, from Eq. (14.17)

$$\varepsilon_a = \varepsilon_{\mathrm{I}} \cos^2\theta + \varepsilon_{\mathrm{II}} \sin^2\theta \tag{14.26}$$

in which ε_n has become ε_a, ε_x has become ε_{I}, ε_y has become $\varepsilon_{\mathrm{II}}$ and γ_{xy} is zero since the x and y directions have become principal directions. This situation is equivalent, as far as ε_a, ε_{I} and $\varepsilon_{\mathrm{II}}$ are concerned, to the strains acting on a triangular element as shown in Fig. 14.17(c). Rewriting Eq. (14.26) we have

$$\varepsilon_a = \frac{\varepsilon_{\mathrm{I}}}{2}(1 + \cos 2\theta) + \frac{\varepsilon_{\mathrm{II}}}{2}(1 - \cos 2\theta)$$

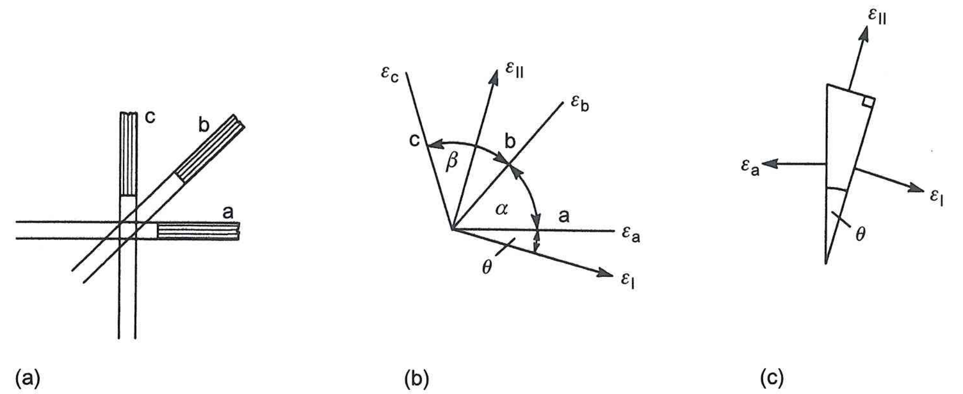


FIGURE 14.17

Electrical resistance strain gauge measurement.

or

$$\varepsilon_a = \frac{1}{2}(\varepsilon_I + \varepsilon_{II}) + \frac{1}{2}(\varepsilon_I - \varepsilon_{II}) \cos 2\theta \qquad (14.27)$$

Similarly

$$\varepsilon_b = \frac{1}{2}(\varepsilon_I + \varepsilon_{II}) + \frac{1}{2}(\varepsilon_I - \varepsilon_{II}) \cos 2(\theta + \alpha) \qquad (14.28)$$

and

$$\varepsilon_c = \frac{1}{2}(\varepsilon_I + \varepsilon_{II}) + \frac{1}{2}(\varepsilon_I - \varepsilon_{II}) \cos 2(\theta + \alpha + \beta) \qquad (14.29)$$

Therefore if ε_a, ε_b and ε_c are measured in given directions, i.e. given angles α and β, then ε_I, ε_{II} and θ are the only unknowns in Eqs (14.27), (14.28) and (14.29).

Having determined the principal strains we obtain the principal stresses using relationships derived in Section 7.8. Thus

$$\varepsilon_I = \frac{1}{E}(\sigma_I - \nu\sigma_{II}) \qquad (14.30)$$

and

$$\varepsilon_{II} = \frac{1}{E}(\sigma_{II} - \nu\sigma_I) \qquad (14.31)$$

Solving Eqs (14.30) and (14.31) for σ_I and σ_{II} we have

$$\sigma_I = \frac{E}{1-\nu^2}(\varepsilon_I + \nu\varepsilon_{II}) \qquad (14.32)$$

and

$$\sigma_{II} = \frac{E}{1-\nu^2}(\varepsilon_{II} + \nu\varepsilon_I) \qquad (14.33)$$

For a 45° rosette $\alpha = \beta = 45°$ and the principal strains may be obtained using the geometry of Mohr's circle of strain. Suppose that the arm 'a' of the rosette is inclined at some unknown angle θ to

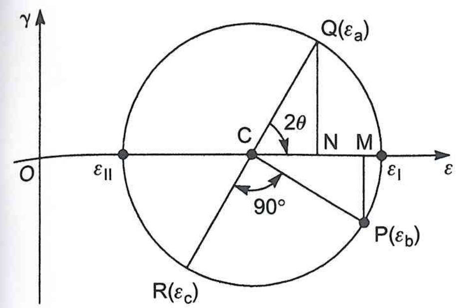


FIGURE 14.18

Mohr's circle of strain for a 45° strain gauge rosette.

the maximum principal strain as in Fig. 14.17(b). Then Mohr's circle of strain is as shown in Fig. 14.18; the shear strains γ_a, γ_b and γ_c do not feature in the discussion and are therefore ignored.

From Fig. 14.18

$$OC = \frac{1}{2}(\varepsilon_a + \varepsilon_c)$$

$$CN = \varepsilon_a - OC = \frac{1}{2}(\varepsilon_a - \varepsilon_c)$$

$$QN = CM = \varepsilon_b - OC = \varepsilon_b - \frac{1}{2}(\varepsilon_a + \varepsilon_c)$$

The radius of the circle is CQ and

$$CQ = \sqrt{CN^2 + QN^2}$$

Hence

$$CQ = \sqrt{\left[\frac{1}{2}(\varepsilon_a - \varepsilon_c)\right]^2 + \left[\varepsilon_b - \frac{1}{2}(\varepsilon_a + \varepsilon_c)\right]^2}$$

which simplifies to

$$CQ = \frac{1}{\sqrt{2}}\sqrt{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_c - \varepsilon_b)^2}$$

Therefore ε_I, which is given by

$$\varepsilon_I = OC + \text{radius of the circle}$$

is

$$\varepsilon_I = \frac{1}{2}(\varepsilon_a + \varepsilon_c) + \frac{1}{\sqrt{2}}\sqrt{(\varepsilon_a - \varepsilon_b)^2 + (\varepsilon_c - \varepsilon_b)^2} \qquad (14.34)$$

Also

$$\varepsilon_{II} = OC - \text{radius of the circle}$$

i.e.

$$\varepsilon_{\mathrm{II}}=\frac{1}{2}(\varepsilon_{\mathrm{a}}+\varepsilon_{\mathrm{c}})-\frac{1}{\sqrt{2}}\sqrt{(\varepsilon_{\mathrm{a}}-\varepsilon_{\mathrm{b}})^2+(\varepsilon_{\mathrm{c}}-\varepsilon_{\mathrm{b}})^2} \qquad (14.35)$$

Finally the angle θ is given by

$$\tan 2\theta=\frac{\mathrm{QN}}{\mathrm{CN}}=\frac{\varepsilon_{\mathrm{b}}-(1/2)(\varepsilon_{\mathrm{a}}+\varepsilon_{\mathrm{c}})}{(1/2)(\varepsilon_{\mathrm{a}}-\varepsilon_{\mathrm{c}})}$$

i.e.

$$\tan 2\theta=\frac{2\varepsilon_{\mathrm{b}}-\varepsilon_{\mathrm{a}}-\varepsilon_{\mathrm{c}}}{\varepsilon_{\mathrm{a}}-\varepsilon_{\mathrm{c}}} \qquad (14.36)$$

A similar approach can be adopted for a 60° rosette.

EXAMPLE 14.7

A shaft of solid circular cross section has a diameter of 50 mm and is subjected to a torque, T, and axial load, P. A rectangular strain gauge rosette attached to the surface of the shaft recorded the following values of strain: $\varepsilon_{\mathrm{a}}=1000\times10^{-6}$, $\varepsilon_{\mathrm{b}}=-200\times10^{-6}$ and $\varepsilon_{\mathrm{c}}=-300\times10^{-6}$ where the gauges 'a' and 'c' are in line with and perpendicular to the axis of the shaft, respectively. If the material of the shaft has a Young's modulus of 70 000 N/mm^2 and a Poisson's ratio of 0.3, calculate the values of T and P.

Substituting the values of ε_{a}, ε_{b} and ε_{c} in Eq. (14.34) we have

$$\varepsilon_{\mathrm{I}}=\frac{10^{-6}}{2}(1000-300)+\frac{10^{-6}}{\sqrt{2}}\sqrt{(1000+200)^2+(-200+300)^2}$$

which gives

$$\varepsilon_{\mathrm{I}}=\frac{10^{-6}}{2}(700+1703)=1202\times10^{-6}$$

It follows from Eq. (14.35) that

$$\varepsilon_{\mathrm{II}}=\frac{10^{-6}}{2}(700-1703)=-502\times10^{-6}$$

Substituting for ε_{I} and $\varepsilon_{\mathrm{II}}$ in Eq. (14.32) we have

$$\sigma_{\mathrm{I}}=\frac{70\,000\times10^{-6}}{1-(0.3)^2}(1202-0.3\times502)=80.9\ \mathrm{N/mm^2}$$

Similarly from Eq. (14.33)

$$\sigma_{\mathrm{II}}=\frac{70\,000\times10^{-6}}{1-(0.3)^2}(-502+0.3\times1202)=-10.9\ \mathrm{N/mm^2}$$

Since $\sigma_y=0$ (note that the axial load produces σ_x only), Eqs (14.8) and (14.9) reduce to

$$\sigma_{\mathrm{I}}=\frac{\sigma_x}{2}+\frac{1}{2}\sqrt{\sigma_x^2+4\tau_{xy}^2} \qquad \text{(i)}$$

and

$$\sigma_{\mathrm{II}}=\frac{\sigma_x}{2}-\frac{1}{2}\sqrt{\sigma_x^2+4\tau_{xy}^2} \qquad \text{(ii)}$$

respectively. Adding Eqs (i) and (ii) we obtain

$$\sigma_{\mathrm{I}}+\sigma_{\mathrm{II}}=\sigma_x$$

Thus

$$\sigma_x=80.9-10.9=70\ \mathrm{N/mm^2}$$

Substituting for σ_x in either of Eq. (i) or (ii) gives

$$\tau_{xy}=29.7\ \mathrm{N/mm^2}$$

For an axial load P

$$\sigma_x=70\ \mathrm{N/mm^2}=\frac{P}{A}=\frac{P}{(\pi/4)\times50^2}\ \text{(Eq. (7.1))}$$

so that

$$P=137.4\ \mathrm{kN}$$

Also for the torque T and using Eq. (11.4) we have

$$\tau_{xy}=29.7\ \mathrm{N/mm^2}=\frac{Tr}{J}=\frac{T\times25}{(\pi/32)\times50^4}$$

which gives

$$T=0.7\ \mathrm{kNm}$$

Note that P could have been found directly in this case from the axial strain ε_{a}. Thus from Eq. (7.8)

$$\sigma_x=E\varepsilon_{\mathrm{a}}=70\,000\times1000\times10^{-6}=70\ \mathrm{N/mm^2}$$

as before.

EXAMPLE 14.8

The thin-walled cantilever box beam shown in Fig. 14.19 has a bar attached to its free end; the bar carries a vertical load W at a distance r from the vertical plane of symmetry. A rectangular strain gauge rosette is attached to the upper cover of the box beam in the vertical plane of symmetry and at a distance of 1 m from the free end. If the readings from the three arms of the rosette are:

$$\varepsilon_{\mathrm{a}}=1200\times10^{-6},\quad \varepsilon_{\mathrm{b}}=200\times10^{-6},\quad \varepsilon_{\mathrm{c}}=-360\times10^{-6}$$

determine the values of W and r. Take $E=200000\ \mathrm{N/mm^2}$ and $\nu=0.3$.

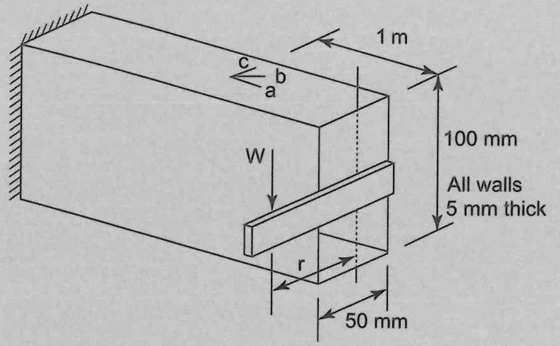


FIGURE 14.19

Cantilever box beam of Ex. 14.8

From Eq. (14.34)

$$\varepsilon_{\mathrm{I}} = \frac{10^{-6}}{2}\left\{(1200 - 360) + \sqrt{\left[2(1200 - 200)^2 + 2(-360 - 200)^2\right]}\right\}$$

which gives

$$\varepsilon_{\mathrm{I}} = 1230.4 \times 10^{-6}$$

Similarly, from Eq. (14.35)

$$\varepsilon_{\mathrm{II}} = -390.4 \times 10^{-6}$$

Then, from Eq. (14.32)

$$\sigma_{\mathrm{I}} = \frac{200000}{1 - (0.3)^2}(1230.4 - 0.3 \times 390.4) \times 10^{-6}$$

which gives

$$\sigma_{\mathrm{I}} = 244.7\ \mathrm{N/mm^2}$$

Similarly, from Eq. (14.33)

$$\sigma_{\mathrm{II}} = -4.7\ \mathrm{N/mm^2}$$

Since, in this case, $\sigma_y = 0$, Eqs. (14.8) and (14.9) reduce to

$$\sigma_{\mathrm{I}} = \frac{\sigma_x}{2} + \frac{1}{2}\sqrt{(\sigma_x{}^2 + 4\tau_{xy}{}^2)} \qquad \text{(i)}$$

and

$$\sigma_{\mathrm{II}} = \frac{\sigma_x}{2} - \frac{1}{2}\sqrt{(\sigma_x{}^2 + 4\tau_{xy}{}^2)} \qquad \text{(ii)}$$

Adding Eqs. (i) and (ii)

$$\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}} = \sigma_x = 244.7 - 4.7 = 240\ \mathrm{N/mm^2}$$

Substituting for σ_x in Eq. (i) or Eq. (ii) gives

$$\tau_{xy} = 33.9\ \mathrm{N/mm^2}$$

From Eq. (9.9)

$$\sigma_x = \frac{W \times 10^3 \times 50}{2(50 \times 5 \times 50^2 + 5 \times 100^3/12)} = 0.024\ W$$

Therefore

$$W = 240/0.024 = 10000\ \mathrm{N} = 10\ \mathrm{kN}$$

From Eq. (11.21)

$$\tau_{xy} = \frac{10 \times 10^3 r}{2 \times 50 \times 100 \times 5} = 33.9$$

which gives

$$r = 169.5\ \mathrm{mm}$$

Note, that as in Ex. 14.7, we could have obtained σ_x directly from the strain gauge reading, that is, from Eq. (7.8)

$$\sigma_x = 200000 \times 1200 \times 10^{-6} = 240\ \mathrm{N/mm^2}$$

However, we would still require a value for either σ_{I} or σ_{II} in order to obtain τ_{xy} so that the saving in computation would not have been significant.

14.10 Theories of elastic failure

The direct stress in a structural member subjected to simple tension or compression is directly proportional to strain up to the yield point of the material (Section 7.7). It is therefore a relatively simple matter to design such a member using the direct stress at yield as the design criterion. However, as we saw in Section 14.3, the direct and shear stresses at a point in a structural member subjected to a complex loading system are not necessarily the maximum values at the point. In such cases it is not clear how failure occurs, so that it is difficult to determine limiting values of load or alternatively to design a structural member for given loads. An obvious method, perhaps, would be to use direct experiment in which the structural member is loaded until deformations are no longer proportional to the applied load; clearly such an approach would be both time-wasting and uneconomical. Ideally a method is required that relates some parameter representing the applied stresses to, say, the yield stress in simple tension which is a constant for a given material.

In Section 14.3 we saw that a complex two-dimensional stress system comprising direct and shear stresses could be represented by a simpler system of direct stresses only, in other words, the principal stresses. The problem is therefore simplified to some extent since the applied loads are now being represented by a system of direct stresses only. Clearly this procedure could be extended to the three-dimensional case so that no matter how complex the loading and the resulting stress system, there would remain at the most just three principal stresses, σ_{I}, σ_{II} and σ_{III}, as shown, for a three-dimensional element, in Fig. 14.20.

It now remains to relate, in some manner, these principal stresses to the yield stress in simple tension, σ_Y, of the material

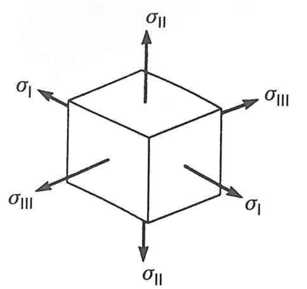


FIGURE 14.20

Reduction of a complex three-dimensional stress system.

Ductile materials

A number of theories of elastic failure have been proposed in the past for ductile materials but experience and experimental evidence have led to all but two being discarded.

Maximum shear stress theory

This theory is usually linked with the names of Tresca and Guest, although it is more widely associated with the former. The theory proposes that:

Failure (i.e. yielding) will occur when the maximum shear stress in the material is equal to the maximum shear stress at failure in simple tension.

For a two-dimensional stress system the maximum shear stress is given in terms of the principal stresses by Eq. (14.12). For a three-dimensional case the maximum shear stress is given by

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} \tag{14.37}$$

where σ_{max} and σ_{min} are the algebraic maximum and minimum principal stresses. At failure in simple tension the yield stress σ_Y is in fact a principal stress and since there can be no direct stress perpendicular to the axis of loading, the maximum shear stress is, therefore, from either of Eqs. (14.12) or (14.37)

$$\tau_{max} = \frac{\sigma_Y}{2} \tag{14.38}$$

Thus the theory proposes that failure in a complex system will occur when

$$\frac{\sigma_{max} - \sigma_{min}}{2} = \frac{\sigma_Y}{2}$$

or

$$\sigma_{max} - \sigma_{min} = \sigma_Y \tag{14.39}$$

Let us now examine stress systems having different relative values of σ_I, σ_{II} and σ_{III}. First suppose that $\sigma_I > \sigma_{II} > \sigma_{III} > 0$. From Eq. (14.39) failure occurs when

$$\sigma_I - \sigma_{III} = \sigma_Y \tag{14.40}$$

Second, suppose that $\sigma_I > \sigma_{II} > 0$ but $\sigma_{III} = 0$. In this case the three-dimensional stress system of Fig. 14.20 reduces to a two-dimensional stress system but *is still acting on a three-dimensional element.* Thus Eq. (14.39) becomes

$$\sigma_I - 0 = \sigma_Y$$

or

$$\sigma_I = \sigma_Y \tag{14.41}$$

Here we see an apparent contradiction of Eq. (14.12) where the maximum shear stress in a two-dimensional stress system is equal to half the difference of σ_I and σ_{II}. However, the maximum shear stress in that case occurs in the plane of the two-dimensional element, i.e. in the plane of σ_I and σ_{II}. In this case we have a three-dimensional element so that the maximum shear stress will lie in the plane of σ_I and σ_{III}.

Finally, let us suppose that $\sigma_I > 0$, $\sigma_{II} < 0$ and $\sigma_{III} = 0$. Again we have a two-dimensional stress system acting on a three-dimensional element but now σ_{II} is a compressive stress and algebraically less than σ_{III}. Thus Eq. (14.39) becomes

$$\sigma_I - \sigma_{II} = \sigma_Y \tag{14.42}$$

Shear strain energy theory

This particular theory of elastic failure was established independently by von Mises, Maxwell and Hencky but is now generally referred to as the von Mises criterion. The theory proposes that:

Failure will occur when the shear or distortion strain energy in the material reaches the equivalent value at yielding in simple tension.

In 1904 Huber proposed that the total strain energy, U_t, of an element of material could be regarded as comprising two separate parts: that due to change in volume and that due to change in shape. The former is termed the volumetric strain energy, U_v, the latter the distortion or shear strain energy, U_s. Thus

$$U_t = U_v + U_s \tag{14.43}$$

Since it is relatively simple to determine U_t and U_v, we obtain U_s by transposing Eq. (14.43). Hence

$$U_s = U_t - U_v \tag{14.44}$$

Initially, however, we shall demonstrate that the deformation of an element of material may be separated into change of volume and change in shape.

The principal stresses σ_I, σ_{II} and σ_{III} acting on the element of Fig. 14.20 may be written as

$$\sigma_I = \frac{1}{3}(\sigma_I + \sigma_{II} + \sigma_{III}) + \frac{1}{3}(2\sigma_I - \sigma_{II} - \sigma_{III})$$

$$\sigma_{II} = \frac{1}{3}(\sigma_I + \sigma_{II} + \sigma_{III}) + \frac{1}{3}(2\sigma_{II} - \sigma_I - \sigma_{III})$$

$$\sigma_{III} = \frac{1}{3}(\sigma_I + \sigma_{II} + \sigma_{III}) + \frac{1}{3}(2\sigma_{III} - \sigma_{II} - \sigma_I)$$

or

$$\left.\begin{aligned} \sigma_I &= \bar{\sigma} + \sigma_I^1 \\ \sigma_{II} &= \bar{\sigma} + \sigma_{II}^1 \\ \sigma_{III} &= \bar{\sigma} + \sigma_{III}^1 \end{aligned}\right\} \tag{14.45}$$

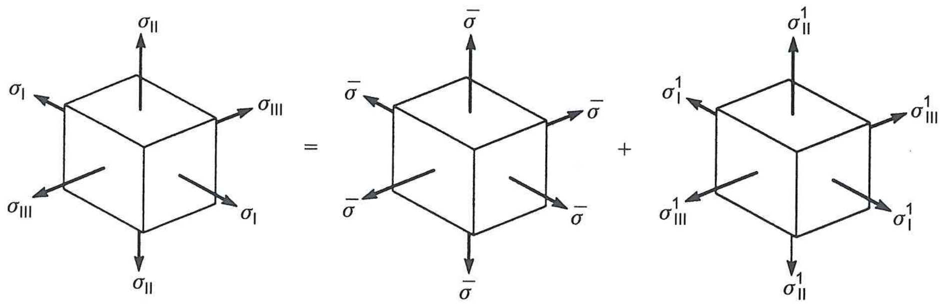


FIGURE 14.21
Representation of principal stresses as volumetric and distortional stresses.

Thus the stress system of Fig. 14.20 may be represented as the sum of two separate stress systems as shown in Fig. 14.21. The $\bar{\sigma}$ stress system is clearly equivalent to a hydrostatic or volumetric stress which will produce a change in volume but not a change in shape. The effect of the σ^1 stress system may be determined as follows. Adding together Eqs (14.45) we obtain

$$\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}} + \sigma_{\mathrm{III}} = 3\bar{\sigma} + \sigma_{\mathrm{I}}^1 + \sigma_{\mathrm{II}}^1 + \sigma_{\mathrm{III}}^1$$

but

$$\bar{\sigma} = \frac{1}{3}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}} + \sigma_{\mathrm{III}})$$

so that

$$\sigma_{\mathrm{I}}^1 + \sigma_{\mathrm{II}}^1 + \sigma_{\mathrm{III}}^1 = 0 \tag{14.46}$$

From the stress–strain relationships of Section 7.8 we have

$$\left.\begin{aligned} \varepsilon_{\mathrm{I}}^1 &= \frac{\sigma_{\mathrm{I}}^1}{E} - \frac{v}{E}(\sigma_{\mathrm{II}}^1 + \sigma_{\mathrm{III}}^1) \\ \varepsilon_{\mathrm{II}}^1 &= \frac{\sigma_{\mathrm{II}}^1}{E} - \frac{v}{E}(\sigma_{\mathrm{I}}^1 + \sigma_{\mathrm{III}}^1) \\ \varepsilon_{\mathrm{III}}^1 &= \frac{\sigma_{\mathrm{III}}^1}{E} - \frac{v}{E}(\sigma_{\mathrm{I}}^1 + \sigma_{\mathrm{II}}^1) \end{aligned}\right\} \tag{14.47}$$

The volumetric strain ε_{v} corresponding to ε_{v} σ_{I}^1, σ_{II}^1, and σ_{III}^1 is equal to the sum of the linear strains. Thus from Eqs (14.47)

$$\varepsilon_{\mathrm{v}} = \varepsilon_{\mathrm{I}}^1 + \varepsilon_{\mathrm{II}}^1 + \varepsilon_{\mathrm{III}}^1 = \frac{(1-2v)}{E}(\sigma_{\mathrm{I}}^1 + \sigma_{\mathrm{II}}^1 + \sigma_{\mathrm{III}}^1)$$

which, from Eq. (14.46), gives

$$\varepsilon_{\mathrm{v}} = 0$$

It follows that σ_{I}^1, σ_{II}^1 and σ_{III}^1 produce no change in volume but only change in shape. We have therefore successfully divided the σ_{I}, σ_{II}, σ_{III} stress system into stresses ($\bar{\sigma}$) producing changes in volume and stresses (σ^1) producing changes in shape.

In Section 7.10 we derived an expression for the strain energy, U, of a member subjected to a direct stress, σ (Eq. (7.30)), i.e.

$$U = \frac{1}{2} \times \frac{\sigma^2}{E} \times \text{volume}$$

This equation may be rewritten

$$U = \frac{1}{2} \times \sigma \times \varepsilon \times \text{volume}$$

since $E = \sigma/\varepsilon$. The strain energy per unit volume is then $\sigma\varepsilon/2$. Thus for a three-dimensional element subjected to a stress $\bar{\sigma}$ on each of its six faces the strain energy in one direction is

$$\frac{1}{2}\bar{\sigma}\,\bar{\varepsilon}$$

where $\bar{\varepsilon}$ is the strain due to $\bar{\sigma}$ in each of the three directions. The total or volumetric strain energy per unit volume, U_{v}, of the element is then given by

$$U_{\mathrm{v}} = 3\left(\frac{1}{2}\bar{\sigma}\,\bar{\varepsilon}\right)$$

or, since

$$\bar{\varepsilon} = \frac{\bar{\sigma}}{E} - 2v\frac{\bar{\sigma}}{E} = \frac{\bar{\sigma}}{E}(1-2v)$$

$$U_{\mathrm{v}} = \frac{1}{2}\bar{\sigma}\frac{3\bar{\sigma}}{E}(1-2v) \tag{14.48}$$

But

$$\bar{\sigma} = \frac{1}{3}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}} + \sigma_{\mathrm{III}})$$

so that Eq. (14.48) becomes

$$U_{\mathrm{v}} = \frac{(1-2v)}{6E}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}} + \sigma_{\mathrm{III}})^2 \tag{14.49}$$

By a similar argument the total strain energy per unit volume, U_{t}, of an element subjected to stresses σ_{I}, σ_{II} and σ_{III} is

$$U_{\mathrm{t}} = \frac{1}{2}\sigma_{\mathrm{I}}\varepsilon_{\mathrm{I}} + \frac{1}{2}\sigma_{\mathrm{II}}\varepsilon_{\mathrm{II}} + \frac{1}{2}\sigma_{\mathrm{III}}\varepsilon_{\mathrm{III}} \tag{14.50}$$

where

$$\left.\begin{aligned} \varepsilon_{\mathrm{I}} &= \frac{\sigma_{\mathrm{I}}}{E} - \frac{v}{E}(\sigma_{\mathrm{II}} + \sigma_{\mathrm{III}}) \\ \varepsilon_{\mathrm{II}} &= \frac{\sigma_{\mathrm{II}}}{E} - \frac{v}{E}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{III}}) \\ \varepsilon_{\mathrm{III}} &= \frac{\sigma_{\mathrm{III}}}{E} - \frac{v}{E}(\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}}) \end{aligned}\right\} \quad \text{(see Eq. (14.47))} \tag{14.51}$$

Substituting for ε_I, etc. in Eq. (14.50) and then for U_v from Eq. (14.49) and U_t in Eq. (14.44) we have

$$U_s = \frac{1}{2E}\left[\sigma_I^2 + \sigma_{II}^2 + \sigma_{III}^2 - 2\nu(\sigma_I\sigma_{II} + \sigma_{II}\sigma_{III} + \sigma_{III}\sigma_I) - \frac{(1-2\nu)}{6E}(\sigma_I + \sigma_{II} + \sigma_{III})^2\right]$$

which simplifies to

$$U_s = \frac{(1+\nu)}{6E}\left[(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2\right]$$

per unit volume.

From Eq. (7.21)

$$E = 2G(1+\nu)$$

Thus

$$U_s = \frac{1}{12G}\left[(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2\right] \tag{14.52}$$

The shear or distortion strain energy per unit volume at failure in simple tension corresponds to $\sigma_I = \sigma_Y$, $\sigma_{II} = \sigma_{III} = 0$. Hence from Eq. (14.52)

$$U_s \text{ (at failure in simple tension)} = \frac{\sigma_Y^2}{6G} \tag{14.53}$$

According to the von Mises criterion, failure occurs when U_s, given by Eq. (14.52), reaches the value of U_s, given by Eq. (14.53), i.e. when

$$(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2 = 2\sigma_Y^2 \tag{14.54}$$

For a two-dimensional stress system in which $\sigma_{III} = 0$, Eq. (14.54) becomes

$$\sigma_I^2 + \sigma_{II}^2 - \sigma_I\sigma_{II} = \sigma_Y^2 \tag{14.55}$$

Design application

Codes of Practice for the use of structural steel in building use the von Mises criterion for a two-dimensional stress system (Eq. (14.55)) in determining an equivalent allowable stress for members subjected to bending and shear. Thus if σ_x and τ_{xy} are the direct and shear stresses, respectively, at a point in a member subjected to bending and shear, then the principal stresses at the point are, from Eqs (14.8) and (14.9)

$$\sigma_I = \frac{\sigma_x}{2} + \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xy}^2}$$

$$\sigma_{II} = \frac{\sigma_x}{2} - \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xy}^2}$$

Substituting these expressions in Eq. (14.55) and simplifying we obtain

$$\sigma_Y = \sqrt{\sigma_x^2 + 3\tau_{xy}^2} \tag{14.56}$$

In Codes of Practice σ_Y is termed an equivalent stress and allowable values are given for a series of different structural members.

Yield loci

Equations (14.39) and (14.54) may be plotted graphically for a two-dimensional stress system in which $\sigma_{III} = 0$ and in which it is assumed that the yield stress, σ_Y, is the same in tension and compression.

Figure 14.22 shows the yield locus for the maximum shear stress or Tresca theory of elastic failure. In the first and third quadrants, when σ_I and σ_{II} have the same sign, failure occurs when either $\sigma_I = \sigma_Y$ or $\sigma_{II} = \sigma_Y$ (see Eq. (14.41)) depending on which principal stress attains the value σ_Y first. For example, a structural member may be subjected to loads that produce a given value of σ_{II} ($<\sigma_Y$) and varying values of σ_I. If the loads were increased, failure would occur when σ_I reached the value σ_Y. Similarly for a fixed value of σ_I and varying σ_{II}. In the second and third quadrants where σ_I and σ_{II} have opposite signs, failure occurs when $\sigma_I - \sigma_{II} = \sigma_Y$ or $\sigma_{II} - \sigma_I = \sigma_Y$ (see Eq. (14.42)). Both these equations represent straight lines, each having a gradient of 45° and an intercept on the σ_{II} axis of σ_Y. Clearly all combinations of σ_I and σ_{II} that lie inside the locus will not cause failure, while all combinations of σ_I and σ_{II} on or outside the locus will. Thus the inside of the locus represents elastic conditions while the outside represents plastic conditions. Note that for the purposes of a yield locus, σ_I and σ_{II} are interchangeable.

The shear strain energy (von Mises) theory for a two-dimensional stress system is represented by Eq. (14.55). This equation may be shown to be that of an ellipse whose major and minor axes are inclined at 45° to the axes of σ_I and σ_{II} as shown in Fig. 14.23. It may also be shown that the ellipse passes through the six corners of the Tresca yield locus so that at these points the two theories give identical results. However, for other combinations of σ_I and σ_{II} the Tresca theory predicts failure where the von Mises theory does not so that the Tresca theory is the more conservative of the two.

The value of the yield loci lies in their use in experimental work on the validation of the different theories. Structural members fabricated from different materials may be subjected to a complete range of combinations of σ_I and σ_{II} each producing failure. The results are then plotted on the yield loci and the accuracy of each theory is determined for different materials.

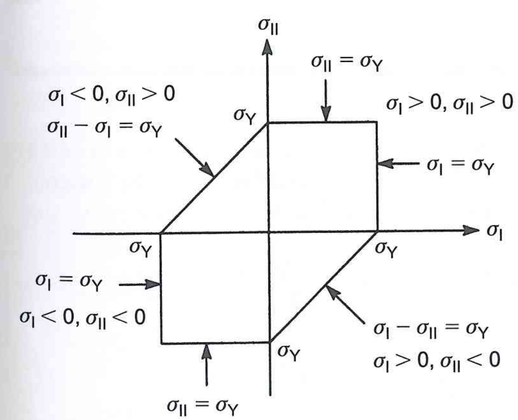


FIGURE 14.22

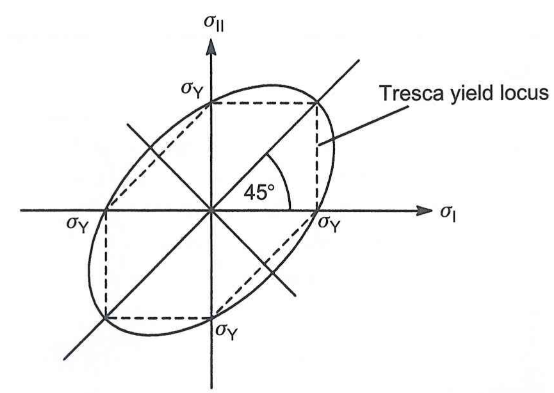


FIGURE 14.23

EXAMPLE 14.9

The state of stress at a point in a structural member is defined by a two-dimensional stress system as follows: $\sigma_x = +140\ \text{N/mm}^2$, $\sigma_y = -70\ \text{N/mm}^2$ and $\tau_{xy} = +60\ \text{N/mm}^2$. If the material of the member has a yield stress in simple tension of 225 N/mm², determine whether or not yielding has occurred according to the Tresca and von Mises theories of elastic failure.

The first step is to determine the principal stresses σ_I and σ_{II}. From Eqs (14.8) and (14.9)

$$\sigma_I = \frac{1}{2}(140-70) + \frac{1}{2}\sqrt{(140+70)^2 + 4\times 60^2}$$

i.e.

$$\sigma_I = 155.9\ \text{N/mm}^2$$

and

$$\sigma_{II} = \frac{1}{2}(140-70) - \frac{1}{2}\sqrt{(140+70)^2 + 4\times 60^2}$$

i.e.

$$\sigma_{II} = -85.9\ \text{N/mm}^2$$

Since σ_{II} is algebraically less than σ_{III} (=0), Eq. (14.42) applies.

Thus

$$\sigma_I - \sigma_{II} = 241.8\ \text{N/mm}^2$$

This value is greater than σ_Y (=225 N/mm²) so that according to the Tresca theory failure has, in fact, occurred.

Substituting the above values of σ_I and σ_{II} in Eq. (14.55) we have

$$(155.9)^2 + (-85.9)^2 - (155.9)(-85.9) = 45\,075.4$$

The square root of this expression is 212.3 N/mm² so that according to the von Mises theory the material has not failed.

EXAMPLE 14.10

The rectangular cross section of a thin-walled box girder (Fig. 14.24) is subjected to a bending moment of 250 kN m and a torque of 200 kN m. If the allowable equivalent stress for the material of the box girder is 180 N/mm², determine whether or not the design is satisfactory using the requirement of Eq. (14.56).

The maximum shear stress in the cross section occurs in the vertical walls of the section and is given by Eq. (11.22), i.e.

$$\tau_{max} = \frac{T_{\max}}{2At_{\min}} = \frac{200\times 10^6}{2\times 500\times 250\times 10} = 80\ \text{N/mm}^2$$

The maximum stress due to bending occurs at the top and bottom of each vertical wall and is given by Eq. (9.9), i.e.

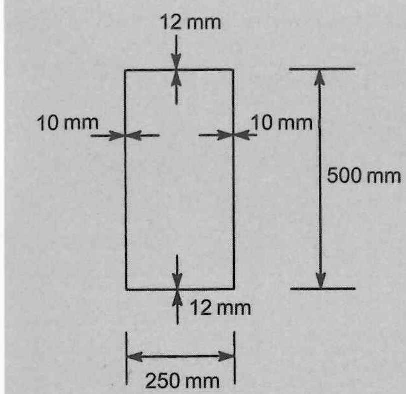


FIGURE 14.24

Box girder beam section of Ex. 14.10.

$$\sigma = \frac{My}{I}$$

where

$$I = 2\times 12\times 250\times 250^2 + \frac{2\times 10\times 500^3}{12} \quad \text{(see Section 9.6)}$$

i.e.

$$I = 583.3\times 10^6\ \text{mm}^4$$

Thus

$$\sigma = \frac{250\times 10^6\times 250}{583.3\times 10^6} = 107.1\ \text{N/mm}^2$$

Substituting these values in Eq. (14.56) we have

$$\sqrt{\sigma_x^2 + 3\tau_{xy}^2} = \sqrt{107.1^2 + 3\times 80^2} = 175.1\ \text{N/mm}^2$$

This equivalent stress is less than the allowable value of 180 N/mm² so that the box girder section is satisfactory.

EXAMPLE 14.11

A beam of rectangular cross section 60 mm × 100 mm is subjected to an axial tensile load of 60 000 N. If the material of the beam fails in simple tension at a stress of 150 N/mm² determine the maximum shear force that can be applied to the beam section in a direction parallel to its longest side using the Tresca and von Mises theories of elastic failure.

The direct stress σ_x due to the axial load is uniform over the cross section of the beam and is given by

$$\sigma_x = \frac{60\,000}{60\times 100} = 10\ \text{N/mm}^2$$

The maximum shear stress τ_{max} occurs at the horizontal axis of symmetry of the beam section nd is, from Eq. (10.7)

$$\tau_{max} = \frac{3}{2} \times \frac{S_y}{60 \times 100} \qquad \text{(i)}$$

Thus from Eqs (14.8) and (14.9)

$$\sigma_I = \frac{10}{2} + \frac{1}{2}\sqrt{10^2 + 4\tau_{max}^2} \qquad \sigma_{II} = \frac{10}{2} - \frac{1}{2}\sqrt{10^2 + 4\tau_{max}^2}$$

r

$$\sigma_I = 5 + \sqrt{25 + \tau_{max}^2} \qquad \sigma_{II} = 5 - \sqrt{25 + \tau_{max}^2} \qquad \text{(ii)}$$

It is clear from the second of Eq. (ii) that σ_{II} is negative since $\left|\sqrt{25 + \tau_{max}^2}\right| > 5$. Thus in the resca theory Eq. (14.42) applies and

$$\sigma_I - \sigma_{II} = 2\sqrt{25 + \tau_{max}^2} = 150 \text{ N/mm}^2$$

rom which

$$\tau_{max} = 74.8 \text{ N/mm}^2$$

Thus from Eq. (i)

$$S_y = 299.3 \text{ kN}$$

Now substituting for σ_I and σ_{II} in Eq. (14.55) we have

$$\left(5+\sqrt{25+\tau_{max}^2}\right)^2 + \left(5-\sqrt{25+\tau_{max}^2}\right)^2 - \left(5+\sqrt{25+\tau_{max}^2}\right)\left(5-\sqrt{25+\tau_{max}^2}\right) = 150^2$$

vhich gives

$$\tau_{max} = 86.4 \text{ N/mm}^2$$

Again from Eq. (i)

$$S_y = 345.6 \text{ kN}$$

ittle materials

hen subjected to tensile stresses brittle materials such as cast iron, concrete and ceramics fracture at a lue of stress very close to the elastic limit with little or no permanent yielding on the planes of maxi- ım shear stress. In fact the failure plane is generally flat and perpendicular to the axis of loading, like ductile materials which have failure planes inclined at approximately 45° to the axis of loading; the latter case failure occurs on planes of maximum shear stress (see Sections 8.3 and 14.2). This uld suggest, therefore, that shear stresses have no effect on the failure of brittle materials and that a ect relationship exists between the principal stresses at a point in a brittle material subjected to a mplex loading system and the failure stress in simple tension or compression. This forms the basis

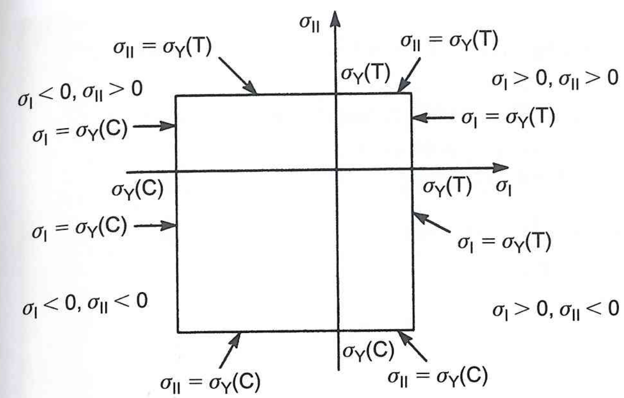


FIGURE 14.25

Yield locus for a brittle material.

Maximum normal stress theory

This theory, frequently attributed to Rankine, states that:

Failure occurs when one of the principal stresses reaches the value of the yield stress in simple tension or compression.

For most brittle materials the yield stress in tension is very much less than the yield stress in compression, e.g. for concrete σ_Y (compression) is approximately 20 σ_Y (tension). Thus it is essential in any particular problem to know which of the yield stresses is achieved first.

Suppose that a brittle material is subjected to a complex loading system which produces principal stresses σ_I, σ_{II} and σ_{III} as in Fig. 14.20. Thus for $\sigma_I > \sigma_{II} > \sigma_{III} > 0$ failure occurs when

$$\sigma_I = \sigma_Y \quad \text{(tension)} \qquad (14.57)$$

Alternatively, for $\sigma_I > \sigma_{II} > 0$, $\sigma_{III} < 0$ and $\sigma_I < \sigma_Y$ (tension) failure occurs when

$$\sigma_{III} = \sigma_Y \quad \text{(compression)} \qquad (14.58)$$

and so on.

A yield locus may be drawn for the two-dimensional case, as for the Tresca and von Mises theories of failure for ductile materials, and is shown in Fig. 14.25. Note that since the failure stress in tension, σ_Y(T), is generally less than the failure stress in compression, σ_Y(C), the yield locus is not symmetrically arranged about the σ_I and σ_{II} axes. Again combinations of stress corresponding to points inside the locus will not cause failure, whereas combinations of σ_I and σ_{II} on or outside the locus will.

EXAMPLE 14.12

A concrete beam has a rectangular cross section 250 mm × 500 mm and is simply supported over a span of 4 m. Determine the maximum mid-span concentrated load the beam can carry if the failure stress in simple tension of concrete is 1.5 N/mm². Neglect the self-weight of the beam.

If the central concentrated load is W N the maximum bending moment occurs at mid-span and is

$$\frac{4W}{4} = W \text{ Nm} \quad \text{(see Ex. 3.7)}$$

The maximum direct tensile stress due to bending occurs at the soffit of the beam and is

$$\sigma = \frac{W \times 10^3 \times 250 \times 10}{250 \times 500^3} = W \times 9.6 \times 10^{-5}\ \text{N/mm}^2\ \text{(Eq.9.9)}$$

At this point the maximum principal stress is, from Eq. (14.8)

$$\sigma_{\text{I}} = W \times 9.6 \times 10^{-5}\ \text{N/mm}^2$$

Thus from Eq. (14.57) the maximum value of *W* is given by

$$\sigma_{\text{I}} = W \times 9.6 \times 10^{-5} = \sigma_Y(\text{tension}) = 1.5\ \text{N/mm}^2$$

from which $W = 15.6$ kN.

The maximum shear stress occurs at the horizontal axis of symmetry of the beam section over each support and is, from Eq. (10.7)

$$\tau_{\max} = \frac{3}{2} \times \frac{W/2}{250 \times 500}$$

i.e.

$$\tau_{\max} = W \times 0.6 \times 10^{-5}\ \text{N/mm}^2$$

Again, from Eq. (14.8), the maximum principal stress is

$$\sigma_{\text{I}} = W \times 9.6 \times 10^{-5}\ \text{N/mm}^2 = \sigma_Y(\text{tension}) = 1.5\ \text{N/mm}^2$$

from which

$$W = 250\ \text{kN}$$

Thus the maximum allowable value of *W* is 15.6 kN.

PROBLEMS

P.14.1 At a point in an elastic material there are two mutually perpendicular planes, one of which carries a direct tensile stress of 50 N/mm² and a shear stress of 40 N/mm² while the other plane is subjected to a direct compressive stress of 35 N/mm² and a complementary shear stress of 40 N/mm². Determine the principal stresses at the point, the position of the planes on which they act and the position of the planes on which there is no direct stress.

Ans. $\sigma_{\text{I}} = 65.9$ N/mm², $\theta = -21.6°$ $\sigma_{\text{II}} = -50.9$ N/mm², $\theta = -111.6°$.
No direct stress on planes at 27.1° and 117.1° to the plane on which the 50 N/mm² stress acts.

P.14.2 One of the principal stresses in a two-dimensional stress system is 139 N/mm² acting on a plane A. On another plane B normal and shear stresses of 108 and 62 N/mm², respectively, act. Determine

a. the angle between the planes A and B,
b. the other principal stress,
c. the direct stress on the plane perpendicular to plane B.

Ans. (a) 26°34′, (b) −16 N/mm², (c) 15 N/mm²

P.14.3 The state of stress at a point in a structural member may be represented by a two-dimensional stress system in which $\sigma_x = 100$ N/mm², $\sigma_y = -80$ N/mm² and $\tau_{xy} = 45$ N/mm². Determine the direct stress on a plane inclined at 60° to the positive direction of σ_x and also the principal stresses. Calculate also the inclination of the principal planes to the plane on which σ_x acts. Verify your answers by a graphical method.

Ans. $\sigma_n = 16$ N/mm² $\sigma_{\text{I}} = 110.6$ N/mm² $\sigma_{\text{II}} = -90.6$ N/mm² $\theta = -13.3°$ and $-103.3°$.

P.14.4 Determine the normal and shear stress on the plane AB shown in Fig. P.14.4 when

i. $\alpha = 60°$, $\sigma_x = 54$ N/mm², $\sigma_y = 30$ N/mm², $\tau_{xy} = 5$ N/mm²;
ii. $\alpha = 120°$, $\sigma_x = -60$ N/mm², $\sigma_y = -36$ N/mm², $\tau_{xy} = 5$ N/mm².

Ans. (i) $\sigma_n = 52.3$ N/mm², $\tau = 7.9$ N/mm²;
(ii) $\sigma_n = -58.3$ N/mm², $\tau = 7.9$ N/mm².

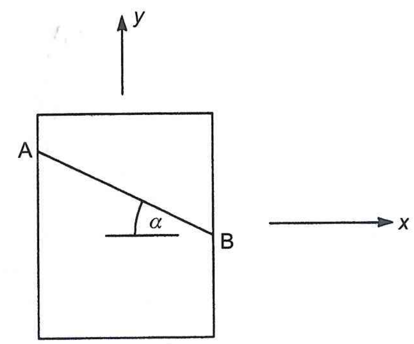


FIGURE P.14.4

P.14.5 A shear stress τ_{xy} acts in a two-dimensional field in which the maximum allowable shear stress is denoted by $\tau_{\max}$ and the major principal stress by σ_{I}. Derive, using the geometry of Mohr's circle of stress, expressions for the maximum values of direct stress which may be applied to the *x* and *y* planes in terms of the parameters given.

Ans.

$$\sigma_x = \sigma_{\text{I}} - \tau_{max} + \sqrt{\tau_{max}^2 - \tau_{xy}^2} \quad \sigma_y = \sigma_{\text{I}} - \tau_{max} - \sqrt{\tau_{max}^2 - \tau_{xy}^2}.$$

P.14.6 In an experimental determination of principal stresses a cantilever of hollow circular cross section is subjected to a varying bending moment and torque; the internal and external diameters of the cantilever are 40 and 50 mm, respectively. For a given loading condition the bending moment and torque at a particular section of the cantilever are 100 and 50 N m, respectively. Calculate the maximum and minimum principal stresses at a point on the outer surface of the cantilever at this section where the direct stress produced by the bending moment is tensile. Determine also the maximum shear stress at the point and the inclination of the principal stresses to the axis of the cantilever.

The experimental values of principal stress are estimated from readings obtained from a 45° strain gauge rosette aligned so that one of its three arms is parallel to and another perpendicular to the axis of the cantilever. For the loading condition of zero torque and

Ans. $\sigma_{\mathrm{I}} = 14.6$ N/mm^2 $\sigma_{\mathrm{II}} = -0.8$ N/mm^2
$\tau_{max} = 7.7$ N/mm^2 $\theta = -13.3°$ and $-103.3°$.

P.14.7 A thin-walled cylinder has an internal diameter of 1200 mm and has walls 1.2 mm thick. It is subjected to an internal pressure of 0.7 N/mm^2 and a torque, about its longitudinal axis, of 500 kN m. Determine the principal stresses at a point in the wall of the cylinder and also the maximum shear stress.

Ans. 466.4 N/mm^2, 58.6 N/mm^2, 203.9 N/mm^2.

P.14.8 A rectangular piece of material is subjected to tensile stresses of 83 and 65 N/mm^2 on mutually perpendicular faces. Find the strain in the direction of each stress and in the direction perpendicular to both stresses. Determine also the maximum shear strain in the plane of the stresses, the maximum shear stress and their directions. Take $E = 200\ 000$ N/mm^2 and $v = 0.3$.

Ans. 3.18×10^{-4},2.01×10^{-4}, -2.22×10^{-4}, $\gamma_{max} = 1.17 \times 10^{-4}$,
$\tau_{max} = 9.0$ N/mm^2 at 45° to the direction of the given stresses.

P.14.9 At a particular point in a structural member a direct tensile stress of 60 N/mm^2 exists on a plane perpendicular to the longitudinal axis of the member while a direct compressive stress of 40 N/mm^2 occurs on a plane parallel to this axis; in addition shear and complementary shear stresses of 50 N/mm^2 act on these planes. If Young's modulus E is 200000 N/mm^2 and Poisson's ratio ν is 0.3 calculate the direct and shear strains on these planes and hence, using a graphical method, determine the principal strains at the point, the maximum shear strain and the angle the plane of maximum principal strain makes with the longitudinal axis of the member. Finally, calculate the principal stresses at the point.

Ans. $\varepsilon_x = 3.6 \times 10^{-4}$, $\varepsilon_y = -2.9 \times 10^{-4}$, $\gamma_{xy} = 6.5 \times 10^{-4}$
$\varepsilon_{\mathrm{I}} = 4.95 \times 10^{-4}$, $\varepsilon_{\mathrm{II}} = -4.25 \times 10^{-4}$, $\gamma_{max} = 9.2 \times 10^{-4}$, 67.5°
$\sigma_{\mathrm{I}} = 80.8$ N/mm^2, $\sigma_{\mathrm{II}} = -60.8$ N/mm^2.

P.14.10 The thin-walled cantilever box beam shown in Fig. P.14.10 carries a vertically downward load of 100 kN at its free end. Calculate the principal strains at the point A which lies at the edge of the top cover of the beam at the built-in end. Take $E = 200000$ N/mm^2 and $\nu = 0.3$.

Ans. $\varepsilon_{\mathrm{I}} = 9.33 \times 10^{-4}$, $\varepsilon_{\mathrm{II}} = -2.87 \times 10^{-4}$.

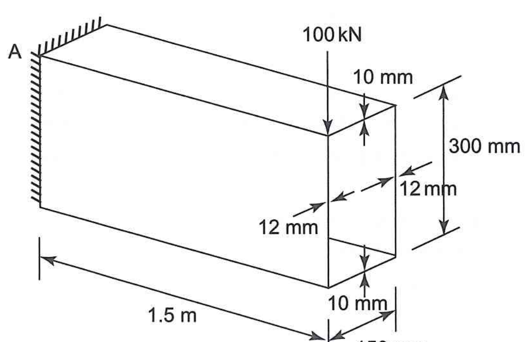


FIGURE P.14.10

P.14.11 A cantilever beam of length 2 m has a rectangular cross section 100 mm wide and 200 mm

distributed load of intensity w. A rectangular strain gauge rosette attached to a vertical side of the beam at the built-in end and in the neutral plane of the beam recorded the following values of strain: $\varepsilon_a = 1000 \times 10^{-6}$, ε_b, $= 100 \times 10^{-6}$, $\varepsilon_c = -300 \times 10^{-6}$. The arm 'a' of the rosette is aligned with the longitudinal axis of the beam while the arm 'c' is perpendicular to the longitudinal axis.

Calculate the value of Poisson's ratio, the principal strains at the point and hence the values of P and w. Young's modulus, $E = 200\ 000$ N/mm^2.

Ans. $P = 4000$ kN $w = 255.3$ kN/m.

P.14.12 A beam has a rectangular thin-walled box section 50 mm wide by 100 mm deep and has walls 2 mm thick. At a particular section the beam carries a bending moment M and a torque T. A rectangular strain gauge rosette positioned on the top horizontal wall of the beam at this section recorded the following values of strain: $\varepsilon_a = 1000 \times 10^{-6}$, $\varepsilon_b = -200 \times 10^{-6}$, $\varepsilon_c = -300 \times 10^{-6}$. If the strain gauge 'a' is aligned with the longitudinal axis of the beam and the strain gauge 'c' is perpendicular to the longitudinal axis, calculate the values of M and T. Take $E = 200\ 000$ N/mm^2 and $v = 0.3$.

Ans. $M = 3333$ Nm $T = 1692$ Nm.

P.14.13 The simply supported beam shown in Fig. P.14.13 carries two symmetrically placed transverse loads, W. A rectangular strain gauge rosette positioned at the point P gave strain readings as follows: $\varepsilon_a = -222 \times 10^{-6}$, $\varepsilon_b = -213 \times 10^{-6}$, $\varepsilon_c = 45 \times 10^{-6}$. Also the direct stress at P due to an external axial compressive load is 7 N/mm^2. Calculate the magnitude of the transverse load. Take $E = 31\ 000$ N/mm^2, $v = 0.2$.

Ans. W = 98.1 kN

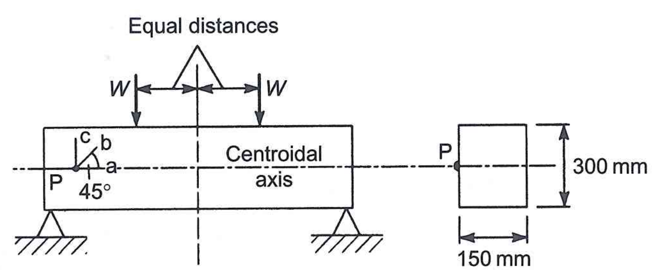


FIGURE P.14.13

P.14.14 A cantilever beam of solid circular cross section is 1 m long and has a diameter of 100 mm. Attached to its free end is a horizontal arm which carries a vertically downward load W at a distance r from the beam's vertical plane of symmetry. On the beam's upper surface, halfway along its length and positioned in its vertical plane of symmetry, is a rectangular strain gauge rosette which gave the following readings for particular values of W and r.

$$\varepsilon_a = 1500 \times 10^{-6},\ \varepsilon_b = -300 \times 10^{-6},\ \varepsilon_c = -450 \times 10^{-6}$$

where the gauges "a" and "c" are aligned with and perpendicular to the axis of the beam respectively. Take $E = 200000$ N/mm^2 and $\nu = 0.3$.

Ans. $W = 58.9$ kN, $r = 423$ mm.

P.14.15 A simply supported beam has a span of 4 m, a rectangular cross section 100 mm wide by 200 mm deep and carries a uniformly distributed load of intensity w N/mm over its complete span. A rectangular strain gauge rosette positioned at mid-span on the upper surface of the beam and in the vertical plane of symmetry recorded the following values of strain:

$$\varepsilon_a = -900 \times 10^{-6}, \ \varepsilon_b = -200 \times 10^{-6}, \ \varepsilon_c = +300 \times 10^{-6}$$

If Young's modulus E is 200000 N/mm^2 calculate the value of Poisson's ratio, the principal stresses at the point and hence the load intensity.

Ans. $\nu = 0.33$, $\sigma_I = 0$, $\sigma_{II} = -180$ N/mm^2, $w = 60$ N/mm.

P.14.16 In a tensile test on a metal specimen having a cross section 20 mm by 10 mm elastic breakdown occurred at a load of 70 000 N.

A thin plate made from the same material is to be subjected to loading such that at a certain point in the plate the stresses are $\sigma_y = -70$ N/mm^2, $\tau_{xy} = 60$ N/mm^2 and σ_x. Determine the maximum allowable values of σ_x using the Tresca and von Mises theories of elastic breakdown.

Ans. 259 N/mm^2 (Tresca) 294 N/mm^2 (von Mises).

P.14.17 A beam of circular cross section is 3000 mm long and is attached at each end to supports which allow rotation of the ends of the beam in the longitudinal vertical plane of symmetry but prevent rotation of the ends in vertical planes perpendicular to the axis of the beam (Fig. P.14.17). The beam supports an offset load of 40 000 N at mid-span.

If the material of the beam suffers elastic breakdown in simple tension at a stress of 145 N/mm^2, calculate the minimum diameter of the beam on the basis of the Tresca and von Mises theories of elastic failure.

Ans. 136 mm (Tresca) 135 mm (von Mises).

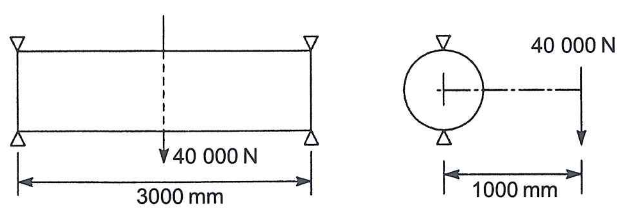


FIGURE P.14.17

P.14.18 A cantilever of circular cross section has a diameter of 150 mm and is made from steel, which, when subjected to simple tension suffers elastic breakdown at a stress of 150 N/mm^2.

The cantilever supports a bending moment and a torque, the latter having a value numerically equal to twice that of the former. Calculate the maximum allowable values of the bending moment and torque on the basis of the Tresca and von Mises theories of elastic failure.

Ans. $M = 22.2$ kNm $T = 44.4$ kN m (Tresca).
$M = 24.9$ kNm $T = 49.8$ kN m (von Mises).

P.14.19 A certain material has a yield stress limit in simple tension of 387 N/mm^2. The yield limit in compression can be taken to be equal to that in tension. The material is subjected to three

Determine the stresses that will cause failure according to the von Mises and Tresca theories of elastic failure.

Ans. Tresca: $\sigma_I = 241.8$ N/mm^2 $\sigma_{II} = 161.2$ N/mm^2 $\sigma_{III} = -145.1$ N/mm^2.
von Mises: $\sigma_I = 264.0$ N/mm^2 $\sigma_{II} = 176.0$ N/mm^2 $\sigma_{III} = -158.4$ N/mm^2.

P.14.20 A thin-walled column has a circular cross section of diameter 200 mm and thickness 5 mm. It carries an axial load P and a torque of 20 kNm. If the material of the column fails in simple tension at a stress of 240 N/mm^2 find the maximum allowable value of P using the Tresca and von Mises theories of elastic failure.

Ans. 640 kN (Tresca), 670.6 kN (von Mises).

P.14.21 A thin-walled box beam has the cross section shown in Fig. P.14.21 and is simply supported over a span of 3 m. The beam carries a concentrated torque of 10 kNm at mid-span together with a vertical uniformly distributed load of intensity w N/mm over its complete span. If the yield stress in simple tension of the material of the beam is 160 N/mm^2 find the maximum allowable value of w using the Tresca and von Mises theories of elastic failure.

Ans. 9.7 N/mm (Tresca), 10.6 N/mm (von Mises).

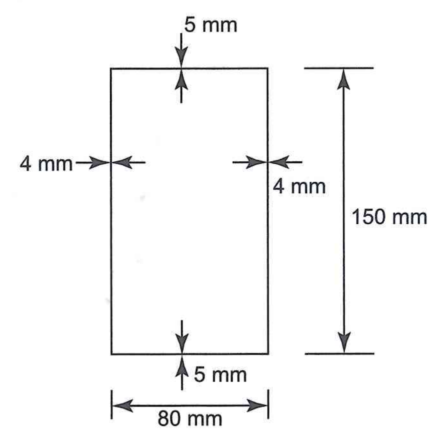


FIGURE P.14.21

P.14.22 The hollow cylinder shown in Fig. P.14.22 is built-in at one end and carries a vertical load of 2 kN offset a distance e from its vertical plane of symmetry. If the material of the cylinder fails in simple tension at a stress of 150 N/mm^2 calculate the maximum allowable value of e using the Tresca and von Mises theories of elastic failure.

Ans. 0.8 m (Tresca), 0.9 m (von Mises).

P.14.23 A thin-walled cylinder of diameter 40 mm is subjected to an internal pressure of 5 N/mm^2 and a torque of 100 Nm. If the material of the cylinder has a yield stress in simple tension of 150 N/mm^2 determine the required wall thickness using the Tresca and von Mises theories of elastic failure.

Ans. 0.81 mm (Tresca), 0.74 mm (von Mises).

P.14.24 A hollow cylindrical shaft has an internal diameter of 100 mm, an external diameter of 130 mm and is required to support a bending moment and a torque each having a value of

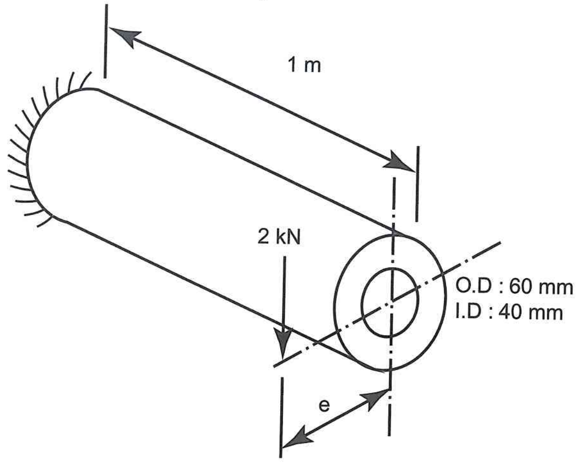


FIGURE P.14.22

20 kNm. If the equivalent stress in the shaft must not exceed 230 N/mm^2 verify that the dimensions of the shaft are satisfactory and also determine the minimum thickness the shaft must have for the same external diameter and loading.

Ans. $\sigma_e = 188.8$ N/mm^2 therefore dimensions are satisfactory.
Minimum thickness = 11.3 mm.

P.14.25 A column has the cross section shown in Fig. P.14.25 and carries a compressive load P parallel to its longitudinal axis. If the failure stresses of the material of the column are 4 and 22 N/mm^2 in simple tension and compression, respectively, determine the maximum allowable value of P using the maximum normal stress theory.

Ans. 634.9 kN.

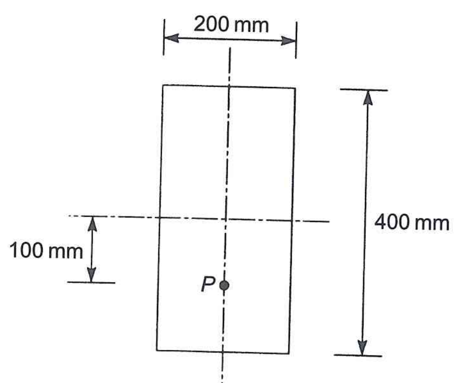


FIGURE P.14.25

P.14.26 A cast iron pipe of external diameter 300 mm has walls 10 mm thick and is required to carry water to a maximum of half its internal depth. If the failure stress in tension of cast iron is 138 N/mm^2 calculate the maximum allowable simply supported span the pipe can have. Take the density of cast iron as 72.3 kN/m^3 and ignore the negligibly small shear stresses.

Ans. 27.0 m.

CHAPTER

Virtual Work and Energy Methods 15

The majority of the structural problems we have encountered so far have involved structures in which the support reactions and the internal force systems are statically determinate. These include beams, trusses, cables and three-pinned arches and, in the case of beams, we have calculated displacements. Some statically indeterminate structures have also been investigated. These include the composite structural members in Section 7.10 and the circular section beams subjected to torsion and supported at each end in Section 11.1. These relatively simple problems were solved using a combination of statical equilibrium and compatibility of displacements. Further, in Section 13.6, a statically indeterminate propped cantilever was analysed using the principle of superposition (Section 3.7) while the support reactions for some cases of fixed beams were determined by combining the conditions of statical equilibrium with the moment-area method (Section 13.3). These methods are perfectly adequate for the comparatively simple problems to which they have been applied. However, other more powerful methods of analysis are required for more complex structures which may possess a high degree of statical indeterminacy. These methods will, in addition, be capable of providing rapid solutions for some statically determinate problems, particularly those involving the calculation of displacements.

The methods fall into two categories and are based on two important concepts; the first, *the principle of virtual work*, is the most fundamental and powerful tool available for the analysis of statically indeterminate structures and has the advantage of being able to deal with conditions other than those in the elastic range, while the second, based on *strain energy*, can provide approximate solutions of complex problems for which exact solutions may not exist. The two methods are, in fact, equivalent in some cases since, although the governing equations differ, the equations themselves are identical.

In modern structural analysis, computer-based techniques are widely used; these include the flexibility and stiffness methods. However, the formulation of, say, stiffness matrices for the elements of a complex structure is based on one of the above approaches, so that a knowledge and understanding of their application is advantageous. We shall examine the flexibility and stiffness methods in Chapter 16 and their role in computer-based analysis.

Other specialist approaches have been developed for particular problems. Examples of these are the slope-deflection method for beams and the moment-distribution method for beams and frames; these will also be described in Chapter 16 where we shall consider statically indeterminate structures. Initially, however, in this chapter, we shall examine the principle of virtual work, the different energy theorems and some of the applications of these two concepts.

15.1 Work

Before we consider the principle of virtual work in detail, it is important to clarify exactly what is meant by *work*. The basic definition of work in elementary mechanics is that 'work is done when a force moves its point of application'. However, we shall require a more exact definition since we shall be concerned with work done by both forces and moments and with the work done by a force when the body on which it acts is given a displacement which is not coincident with the line of action of the force.

Consider the force, F, acting on a particle, A, in Fig. 15.1(a). If the particle is given a displacement, Δ, by some external agency so that it moves to A′ in a direction at an angle α to the line of action of F, the work, W_F, done by F is given by

$$W_F = F(\Delta \cos \alpha) \tag{15.1}$$

or

$$W_F = (F \cos \alpha)\Delta \tag{15.2}$$

Thus we see that the work done by the force, F, as the particle moves from A to A′ may be regarded as either the product of F and the component of Δ in the direction of F (Eq. (15.1)) or as the product of the component of F in the direction of Δ and Δ (Eq. (15.2)).

Now consider the couple (pure moment) in Fig. 15.1(b) and suppose that the couple is given a small rotation of θ radians. The work done by each force F is then $F(a/2)\theta$ so that the total work done, W_C, by the couple is

$$W_C = F\frac{a}{2}\theta + F\frac{a}{2}\theta = Fa\theta$$

It follows that the work done, W_M, by the pure moment, M, acting on the bar AB in Fig. 15.1(c) as it is given a small rotation, θ, is

$$W_M = M\theta \tag{15.3}$$

Note that in the above the force, F, and moment, M, are in position before the displacements take place and are not the cause of them. Also, in Fig. 15.1(a), the component of Δ parallel to the direction of F is in the same direction as F; if it had been in the opposite direction the work done would have been negative. The same argument applies to the work done by the moment, M, where we see in Fig. 15.1(c) that the rotation, θ, is in the same sense as M. Note also that if the displacement, Δ, had been perpendicular to the force, F, no work would have been done by F.

Finally it should be remembered that work is a scalar quantity since it is not associated with direction (in Fig. 15.1(a) the force F does work if the particle is moved in any direction). Thus the work done by a series of forces is the algebraic sum of the work done by each force.

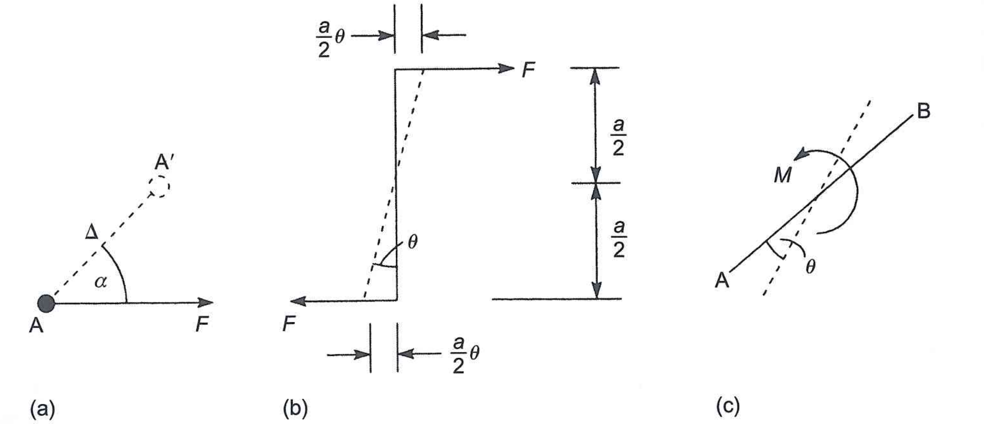


FIGURE 15.1

15.2 Principle of virtual work

The establishment of the principle will be carried out in stages. First we shall consider a particle, then a rigid body and finally a deformable body, which is the practical application we require when analysing structures.

Principle of virtual work for a particle

In Fig. 15.2 a particle, A, is acted upon by a number of concurrent forces, $F_1, F_2, \ldots, F_k, \ldots, F_r$; the resultant of these forces is R. Suppose that the particle is given a small arbitrary displacement, Δ_v, to A′ in some specified direction; Δ_v is an imaginary or *virtual* displacement and is sufficiently small so that the directions of F_1, F_2, etc., are unchanged. Let θ_R be the angle that the resultant, R, of the forces makes with the direction of Δ_v and $\theta_1, \theta_2, \ldots, \theta_k, \ldots, \theta_r$ the angles that $F_1, F_2, \ldots, F_k, \ldots, F_r$ make with the direction of Δ_v, respectively. Then, from either of Eqs (15.1) or (15.2) the total virtual work, W_F, done by the forces F as the particle moves through the virtual displacement, Δ_v, is given by

$$W_F = F_1\Delta_V \cos\theta_1 + F_2\Delta_V \cos\theta_2 + \cdots + F_k\Delta_V \cos\theta_k + \cdots + F_r\Delta_V \cos\theta_r$$

Thus

$$W_F = \sum_{k=1}^{r} F_k\Delta_v \cos\theta_k$$

or, since Δ_v, is a fixed, although imaginary displacement

$$W_F = \Delta_v \sum_{k=1}^{r} F_k \cos\theta_k \tag{15.4}$$

In Eq. (15.4) $\sum_{k=1}^{r} F_k \cos\theta_k$ is the sum of all the components of the forces, F, in the direction of Δ_v, and therefore must be equal to the component of the resultant, R, of the forces, F, in the direction of Δ_v, i.e.

$$W_F = \Delta_v \sum_{k=1}^{r} F_k \cos\theta_k = \Delta_v R \cos\theta_R \tag{15.5}$$

If the particle, A, is in equilibrium under the action of the forces, $F_1, F_2, \ldots, F_k, \ldots, F_r$, the resultant, R, of the forces is zero (Chapter 2). It follows from Eq. (15.5) that the virtual work done by the forces, F, during the virtual displacement, Δ_v, is zero.

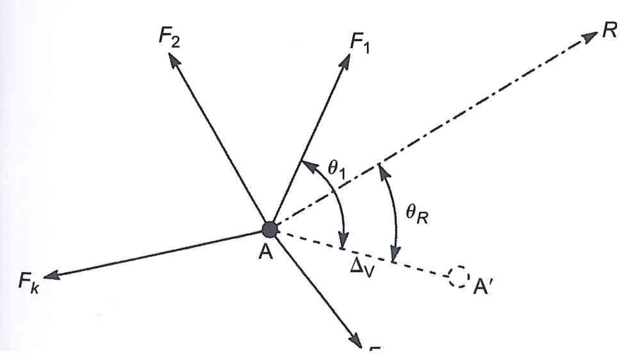


FIGURE 15.2

Virtual work for a system of forces acting on

We can therefore state the *principle of virtual work* for a particle as follows:

If a particle is in equilibrium under the action of a number of forces the total work done by the forces for a small arbitrary displacement of the particle is zero.

It is possible for the total work done by the forces to be zero even though the particle is not in equilibrium if the virtual displacement is taken to be in a direction perpendicular to their resultant, R. We cannot, therefore, state the converse of the above principle unless we specify that the total work done must be zero for *any* arbitrary displacement. Thus:

A particle is in equilibrium under the action of a system of forces if the total work done by the forces is zero for any virtual displacement of the particle.

Note that in the above, Δ_v is a purely imaginary displacement and is not related in any way to the possible displacement of the particle under the action of the forces, F. Δ_v has been introduced purely as a device for setting up the work–equilibrium relationship of Eq. (15.5). The forces, F, therefore remain unchanged in magnitude and direction during this imaginary displacement; this would not be the case if the displacement were real.

Principle of virtual work for a rigid body

Consider the rigid body shown in Fig. 15.3, which is acted upon by a system of external forces, F_1, $F_2, \ldots, F_k, \ldots, F_r$. These external forces will induce internal forces in the body, which may be regarded as comprising an infinite number of particles; on adjacent particles, such as A_1 and A_2, these internal forces will be equal and opposite, in other words self-equilibrating. Suppose now that the rigid body is given a small, imaginary, that is virtual, displacement, Δ_v (or a rotation or a combination of both), in some specified direction. The external and internal forces then do virtual work and the total virtual work done, W_t, is the sum of the virtual work, W_e, done by the external forces and the virtual work, W_i, done by the internal forces. Thus

$$W_t = W_e + W_i \tag{15.6}$$

Since the body is rigid, all the particles in the body move through the same displacement, Δ_v, so that the virtual work done on all the particles is numerically the same. However, for a pair of adjacent particles, such as A_1 and A_2 in Fig. 15.3, the self-equilibrating forces are in opposite directions, which means that the work done on A_1 is opposite in sign to the work done on A_2. Thus the sum of the

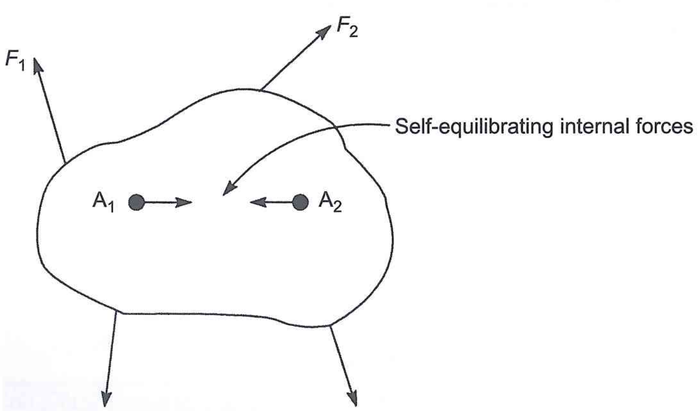


FIGURE 15.3

virtual work done on A_1 and A_2 is zero. The argument can be extended to the infinite number of pairs of particles in the body from which we conclude that the internal virtual work produced by a virtual displacement in a rigid body is zero. Equation (15.6) then reduces to

$$W_t = W_e \tag{15.7}$$

Since the body is rigid and the internal virtual work is therefore zero, we may regard the body as a large particle. It follows that if the body is in equilibrium under the action of a set of forces, F_1, $F_2, \ldots, F_k, \ldots, F_r$, the total virtual work done by the external forces during an arbitrary virtual displacement of the body is zero.

The principle of virtual work is, in fact, an alternative to Eq. (2.10) for specifying the necessary conditions for a system of coplanar forces to be in equilibrium. To illustrate the truth of this we shall consider the calculation of the support reactions in some simple beams.

EXAMPLE 15.1

Calculate the support reactions in the cantilever beam shown in Fig. 15.4(a).

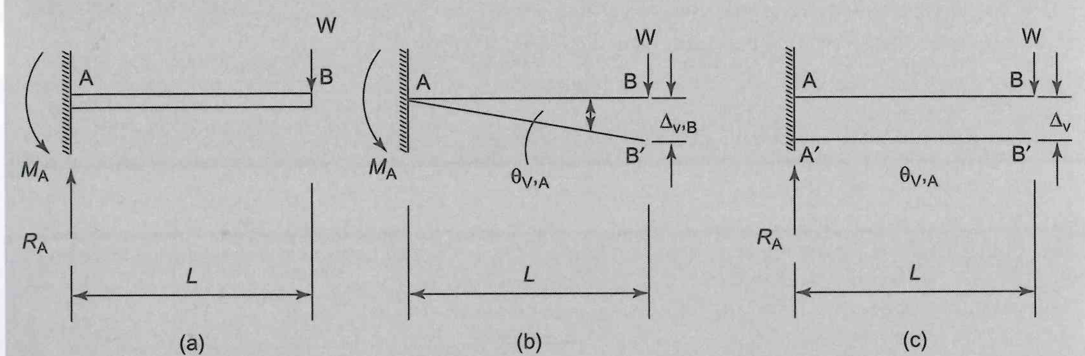


FIGURE 15.4

Beam of Ex. 15.1.

The concentrated load, W, induces a vertical reaction, R_A, and also one of moment, M_A, at A. Suppose that the beam is given a small imaginary, that is virtual, rotation, $\theta_{v,A}$, at A as shown in Fig. 15.4(b). Since we are only concerned here with external forces we may regard the beam as a rigid body so that the beam remains straight and B is displaced to B′. The vertical displacement of B, $\Delta_{v,B}$, is then given by

$$\Delta_{v,B} = \theta_{v,A} L$$

or

$$\theta_{v,A} = \Delta_{v,B}/L \tag{i}$$

The total virtual work, W_t, done by all the forces acting on the beam is given by

$$W_t = W\Delta_{v,B} - M_A\theta_{v,A} \tag{ii}$$

Note that the contribution of M_A to the total virtual work is negative since the assumed direction of M_A is in the opposite sense to the virtual displacement, $\theta_{v,A}$. Note also that there is no linear movement of the beam at A so that R_A does no work. Substituting in Eq. (ii) for $\theta_{v,A}$ from Eq. (i) we have

$$W_t = W\Delta_{v,B} - M_A\Delta_{v,B}/L \qquad \text{(iii)}$$

Since the beam is in equilibrium, $W_t = 0$ from the principle of virtual work. Therefore

$$0 = W\Delta_{v,B} - M_A\Delta_{v,B}/L$$

so that

$$M_A = WL$$

which is the result which would have been obtained from considering the moment equilibrium of the beam about A.

Suppose now that the complete beam is given a virtual displacement, Δ_v, as shown in Fig. 15.4(c). There is no rotation of the beam so that M_A does no work. The total virtual work done is then given by

$$W_t = W\Delta_v - R_A\Delta_v \qquad \text{(iv)}$$

The contribution of R_A is negative since its direction is opposite to that of Δ_v. The beam is in equilibrium so that $W_t = 0$. Therefore, from Eq. (iv)

$$R_A = W$$

which is the result we would have obtained by resolving forces vertically.

EXAMPLE 15.2

Calculate the support reactions in the cantilever beam shown in Fig. 15.5(a).

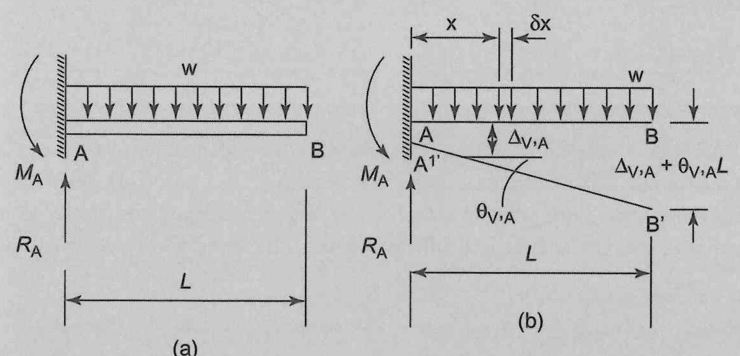


FIGURE 15.5

Beam of Ex. 15.2.

In this case we obtain a solution by simultaneously giving the beam a virtual displacement, $\Delta_{v,A}$, at A and a virtual rotation, $\theta_{v,A}$, at A. The total deflection at B is then $\Delta_{v,A} + \theta_{v,A}L$ and at a distance x from A is $\Delta_{v,A} + \theta_{v,A}x$. Since the beam carries a uniformly distributed load we find the virtual work done by the load by first considering an elemental length, δx, of the load a distance x from A. The load on the element is $w\delta x$ and the virtual work done by this elemental load is given by

$$\delta W = w\delta x(\Delta_{v,A} + \theta_{v,A}x)$$

The total virtual work done on the beam is then

$$W_t = \int_0^L w(\Delta_{v,A} + \theta_{v,A}x)\mathrm{d}x - M_A\theta_{v,A} - R_A\Delta_{v,A}$$

which simplifies to

$$W_t = (wL - R_A)\,\Delta_{v,A} + [(wL^2/2) - M_A]\theta_{v,A} = 0 \qquad \text{(i)}$$

since the beam is in equilibrium. Equation (i) is valid for all values of $\Delta_{v,A}$ and $\theta_{v,A}$ so that

$$wL - R_A = 0 \text{ and } (wL^2/2) - M_A = 0$$

Therefore

$$R_A = wL \text{ and } M_A = wL^2/2$$

which are the results that would have been obtained by resolving forces vertically and taking moments about A.

EXAMPLE 15.3

Calculate the reactions at the built-in end of the cantilever beam shown in Fig. 15.6.

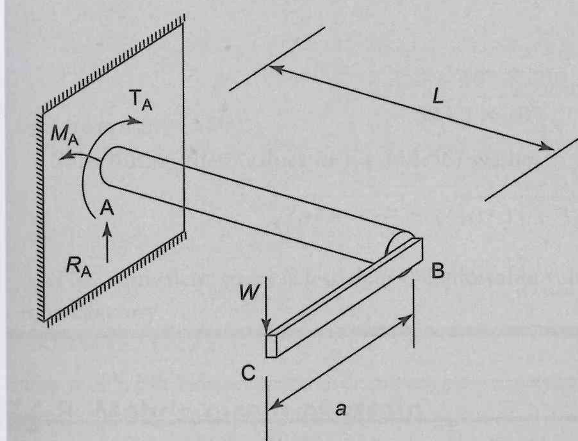


FIGURE 15.6

Beam of Ex. 15.3.

In this example the load, W, produces reactions of vertical force, moment and torque at the built-in end. The vertical reaction and the moment are the same as in Ex. 15.1. To determine the torque reaction we impose a small virtual displacement, $\Delta_{v,C}$, vertically downwards at C. This causes the beam AB to rotate as a rigid body through an angle, $\theta_{v,AB}$, which is given by

$$\theta_{v,AB} = \Delta_{v,C}/a \qquad \text{(i)}$$

Alternatively we could have imposed a small virtual rotation, $\theta_{v,AB}$, on the beam which would have resulted in a virtual displacement of C equal to $a\theta_{v,AB}$: clearly the two approaches produce identical results.

The total virtual work done on the beam is then given by

$$W_t = W\Delta_{v,C} - T_A\theta_{v,AB} = 0 \qquad \text{(ii)}$$

since the beam is in equilibrium. Substituting for $\theta_{v,AB}$ in Eq. (ii) from Eq. (i) we have

$$T_A = Wa$$

which is the result that would have been obtained by considering the statical equilibrium of the beam.

EXAMPLE 15.4

Calculate the support reactions in the simply supported beam shown in Fig. 15.7.

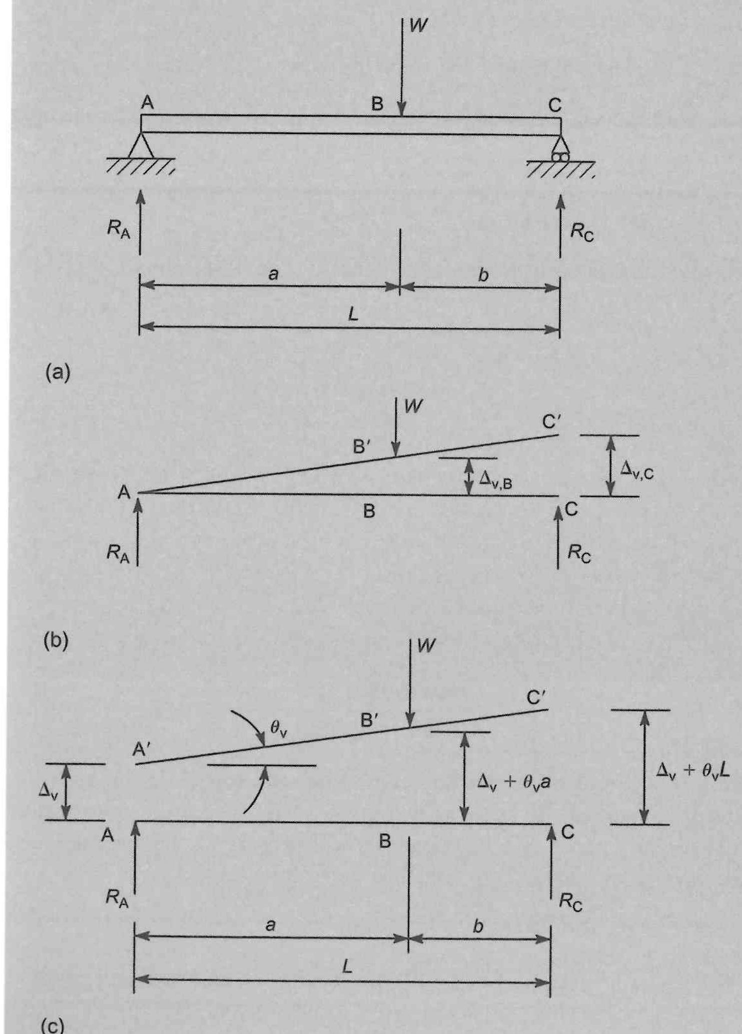


FIGURE 15.7
Use of the principle of virtual work to calculate support reactions.

Only a vertical load is applied to the beam so that only vertical reactions, R_A and R_C, are produced.

Suppose that the beam at C is given a small imaginary, that is a virtual, displacement, $\Delta_{v,C}$, in the direction of R_C as shown in Fig. 15.7(b). Since we are concerned here solely with the *external* forces acting on the beam we may regard the beam as a rigid body. The beam therefore rotates about A so that C moves to C′ and B moves to B′. From similar triangles we see that

$$\Delta_{v,B} = \frac{a}{a+b}\Delta_{v,C} = \frac{a}{L}\Delta_{v,C} \qquad \text{(i)}$$

The total virtual work, W_t, done by all the forces acting on the beam is then given by

$$W_t = R_C\Delta_{v,C} - W\Delta_{v,B} \qquad \text{(ii)}$$

Note that the work done by the load, W, is negative since $\Delta_{v,B}$ is in the opposite direction to its line of action. Note also that the support reaction, R_A, does no work since the beam only rotates about A. Now substituting for $\Delta_{v,B}$ in Eq. (ii) from Eq. (i) we have

$$W_t = R_C\Delta_{v,C} - W\frac{a}{L}\Delta_{v,C} \qquad \text{(iii)}$$

Since the beam is in equilibrium, W_t is zero from the principal of virtual work. Hence, from Eq. (iii)

$$R_C\Delta_{v,C} - W\frac{a}{L}\Delta_{v,C} = 0$$

which gives

$$R_C = W\frac{a}{L}$$

which is the result that would have been obtained from a consideration of the moment equilibrium of the beam about A. R_A follows in a similar manner. Suppose now that instead of the single displacement $\Delta_{v,C}$ the complete beam is given a vertical virtual displacement, Δ_v, together with a virtual rotation, θ_v, about A as shown in Fig. 15.7(c). The total virtual work, W_t, done by the forces acting on the beam is now given by

$$W_t = R_A\Delta_v - W(\Delta_v + a\theta_v) + R_C(\Delta_v + L\theta_v) = 0 \qquad \text{(iv)}$$

since the beam is in equilibrium. Rearranging Eq. (iv)

$$(R_A + R_C - W)\Delta_v + (R_C L - Wa)\theta_v = 0 \qquad \text{(v)}$$

Equation (v) is valid for all values of Δ_v and θ_v so that

$$R_A + R_C - W = 0 \quad R_C L - Wa = 0$$

which are the equations of equilibrium we would have obtained by resolving forces vertically and taking moments about A.

It is not being suggested here that the application of Eq. (2.10) should be abandoned in favour of the principle of virtual work. The purpose of Exs 15.1–15.4 is to illustrate the application of a virtual displacement and the manner in which the principle is used.

Virtual work in a deformable body

In structural analysis we are not generally concerned with forces acting on a rigid body. Structures and structural members deform under load, which means that if we assign a virtual displacement to a particular point in a structure, not all points in the structure will suffer the same virtual displacement as would be the case if the structure were rigid. This means that the virtual work produced by the internal forces is not zero as it is in the rigid body case, since the virtual work produced by the self-equilibrating forces on adjacent particles does not cancel out. The total virtual work produced by applying a virtual displacement to a deformable body acted upon by a system of external forces is therefore given by Eq. (15.6).

If the body is in equilibrium under the action of the external force system then every particle in the body is also in equilibrium. Therefore, from the principle of virtual work, the virtual work done by the forces acting on the particle is zero irrespective of whether the forces are external or internal. It follows that, since the virtual work is zero for all particles in the body, it is zero for the complete body and Eq. (15.6) becomes

$$W_e + W_i = 0 \tag{15.8}$$

Note that in the above argument only the conditions of equilibrium and the concept of work are employed. Thus Eq. (15.8) does not require the deformable body to be linearly elastic (i.e. it need not obey Hooke's law) so that the principle of virtual work may be applied to any body or structure that is rigid, elastic or plastic. The principle does require that displacements, whether real or imaginary, must be small, so that we may assume that external and internal forces are unchanged in magnitude and direction during the displacements. In addition the virtual displacements must be compatible with the geometry of the structure and the constraints that are applied, such as those at a support. The exception is the situation we have in Exs 15.1–15.4 where we apply a virtual displacement at a support. This approach is valid since we include the work done by the support reactions in the total virtual work equation.

Work done by internal force systems

The calculation of the work done by an external force is straightforward in that it is the product of the force and the displacement of its point of application in its own line of action (Eqs (15.1), (15.2) or (15.3)) whereas the calculation of the work done by an internal force system during a displacement is much more complicated. In Chapter 3 we saw that no matter how complex a loading system is, it may be simplified to a combination of up to four load types: axial load, shear force, bending moment and torsion; these in turn produce corresponding internal force systems. We shall now consider the work done by these internal force systems during arbitrary virtual displacements.

Axial force

Consider the elemental length, δx, of a structural member as shown in Fig. 15.8 and suppose that it is subjected to a positive internal force system comprising a normal force (i.e. axial force), N, a shear force, S, a bending moment, M and a torque, T, produced by some external loading system acting on the structure of which the member is part. Note that the face on which the internal forces act is a negative face, see Fig. 3.7. The stress distributions corresponding to these internal forces have been related in previous chapters to an axis system whose origin coincides with the centroid of area of the cross section. We shall, in fact, be using these stress distributions in the derivation of expressions for internal virtual work in linearly elastic structures so that it is logical to assume the same origin of axes here; we shall also assume that the y axis is

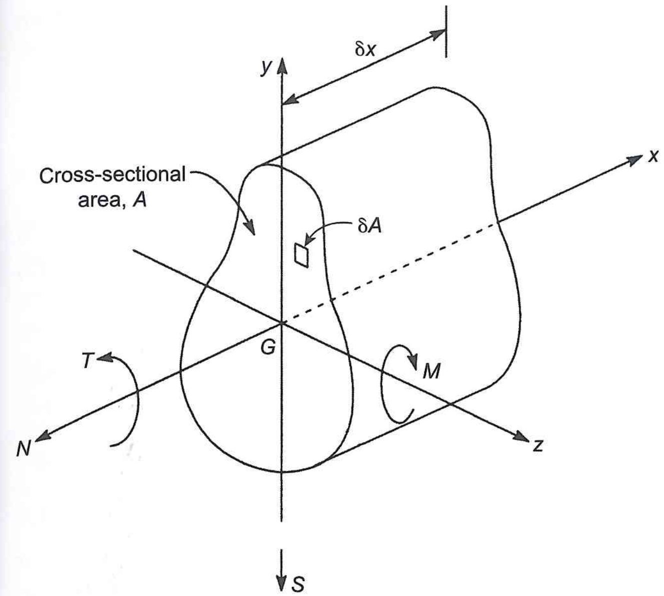


FIGURE 15.8

Virtual work due to internal force system.

The direct stress, σ, at any point in the cross section of the member is given by $\sigma = N/A$ (Eq. (7.1)). Therefore the normal force on the element δA at the point (z, y) is

$$\delta N = \sigma \delta A = \frac{N}{A}\delta A$$

Suppose now that the structure is given an arbitrary virtual displacement which produces a virtual axial strain, ε_v, in the element. The internal virtual work, $\delta w_{i,N}$, done by the axial force on the elemental length of the member is given by

$$\delta w_{i,N} = \int_A \frac{N}{A} \mathrm{d}A \varepsilon_v \delta x$$

which, since $\int_A \mathrm{d}A = A$, reduces to

$$\delta w_{i,N} = N\varepsilon_v \delta x \tag{15.9}$$

In other words, the virtual work done by N is the product of N and the virtual axial displacement of the element of the member. For a member of length L, the virtual work, $w_{i,N}$, done during the arbitrary virtual strain is then

$$w_{i,N} = \int_L N\varepsilon_v \mathrm{d}x \tag{15.10}$$

For a structure comprising a number of members, the total internal virtual work, $W_{i,N}$, done by axial force is the sum of the virtual work of each of the members. Thus

$$w_{i,N} = \sum \int_L N\varepsilon_v \mathrm{d}x \tag{15.11}$$

Note that in the derivation of Eq. (15.11) we have made no assumption regarding the material

However, for a linearly elastic material, i.e. one that obeys Hooke's law (Section 7.7), we can express the virtual strain in terms of an equivalent virtual normal force. Thus

$$\varepsilon_v = \frac{\sigma_v}{E} = \frac{N_v}{EA}$$

Therefore, if we designate the *actual* normal force in a member by N_A, Eq. (15.11) may be expressed in the form

$$w_{i,N} = \sum \int_L \frac{N_A N_v}{EA} dx \tag{15.12}$$

Shear force

The shear force, S, acting on the member section in Fig. 15.8 produces a distribution of vertical shear stress which, as we saw in Section 10.2, depends upon the geometry of the cross section. However, since the element, δA, is infinitesimally small, we may regard the shear stress, τ, as constant over the element. The shear force, δS, on the element is then

$$\delta S = \tau\, \delta A \tag{15.13}$$

Suppose that the structure is given an arbitrary virtual displacement which produces a virtual shear strain, γ_v, at the element. This shear strain represents the angular rotation in a vertical plane of the element $\delta A \times \delta x$ relative to the longitudinal centroidal axis of the member. The vertical displacement at the section being considered is therefore $\gamma_v\, \delta x$. The internal virtual work, $\delta w_{i,S}$, done by the shear force, S, on the elemental length of the member is given by

$$\delta w_{i,S} = \int_A \tau \mathrm{d}A \gamma_v\, \delta x$$

We saw in Section 13.5 that we could assume a uniform shear stress through the cross section of a beam if we allowed for the actual variation by including a form factor, β. Thus the expression for the internal virtual work in the member may be written

$$\delta w_{i,S} = \int_A \beta\left(\frac{S}{A}\right) \mathrm{d}A \gamma_v\, \delta x$$

or

$$\delta w_{i,S} = \beta S \gamma_v\, \delta x \tag{15.14}$$

Hence the virtual work done by the shear force during the arbitrary virtual strain in a member of length L is

$$w_{i,S} = \beta \int_L S \gamma_v\, \mathrm{d}x \tag{15.15}$$

For a linearly elastic member, as in the case of axial force, we may express the virtual shear strain, γ_v, in terms of an equivalent virtual shear force, S_v. Thus, from Section 7.7

$$\gamma_v = \frac{\tau_v}{G} = \frac{S_v}{GA}$$

so that from Eq. (15.15)

$$w_{i,S} = \beta \int_L \frac{S_A S_v}{GA} \mathrm{d}x \tag{15.16}$$

For a structure comprising a number of linearly elastic members the total internal work, $W_{i,S}$, done by the shear forces is

$$W_{i,S} = \sum \beta \int_L \frac{S_A S_v}{GA} \mathrm{d}x \tag{15.17}$$

Bending moment

The bending moment, M, acting on the member section in Fig. 15.8 produces a distribution of direct stress, σ, through the depth of the member cross section. The normal force on the element, δA, corresponding to this stress is therefore $\sigma\, \delta A$. Again we shall suppose that the structure is given a small arbitrary virtual displacement which produces a virtual direct strain, ε_v, in the element $\delta A \times \delta x$. Thus the virtual work done by the normal force acting on the element δA is $\sigma\, \delta A\, \varepsilon_v\, \delta x$. Hence, integrating over the complete cross section of the member we obtain the internal virtual work, $\delta w_{i,M}$, done by the bending moment, M, on the elemental length of member, i.e.

$$\delta w_{i,M} = \int_A \sigma \mathrm{d}A \varepsilon_v\, \delta x \tag{15.18}$$

The virtual strain, ε_v, in the element $\delta A \times \delta x$ is, from Eq. (9.1), given by

$$\varepsilon_v = \frac{y}{R_v}$$

where R_v is the radius of curvature of the member produced by the virtual displacement. Thus, substituting for ε_v in Eq. (15.18), we obtain

$$\delta w_{i,M} = \int_A \sigma \frac{y}{R_v} \mathrm{d}A\, \delta x$$

or, since $\sigma y\, \delta A$ is the moment of the normal force on the element, δA, about the z axis,

$$\delta w_{i,M} = \frac{M}{R_v}\, \delta x$$

Therefore, for a member of length L, the internal virtual work done by an actual bending moment, M_A, is given by

$$w_{i,M} = \int_L \frac{M_A}{R_v} \mathrm{d}x \tag{15.19}$$

In the derivation of Eq. (15.19) no specific stress–strain relationship has been assumed, so that it is applicable to a non-linear system. For the particular case of a linearly elastic system, the virtual curvature $1/R_v$ may be expressed in terms of an equivalent virtual bending moment, M_v, using the relationship of Eq. (9.11), i.e.

$$\frac{1}{R_v} = \frac{M_v}{EI}$$

Substituting for $1/R_v$ in Eq. (15.19) we have

$$w_{i,M} = \int \frac{M_A M_v}{} \mathrm{d}x \tag{15.20}$$

so that for a structure comprising a number of members the total internal virtual work, $W_{i,M}$, produced by bending is

$$W_{i,M} = \sum \int_L \frac{M_A M_v}{EI}\, dx \qquad (15.21)$$

In Chapter 9 we used the suffix 'z' to denote a bending moment in a vertical plane about the z axis (M_z) and the second moment of area of the member section about the z axis (I_z). Clearly the bending moments in Eq. (15.21) need not be restricted to those in a vertical plane; the suffixes are therefore omitted.

Torsion

The internal virtual work, $w_{i,T}$, due to torsion in the particular case of a linearly elastic circular section bar may be found in a similar manner and is given by

$$w_{i,T} = \int_L \frac{T_A T_v}{GI_o}\, dx \qquad (15.22)$$

in which I_o is the polar second moment of area of the cross section of the bar (see Section 11.1). For beams of non-circular cross section, I_o is replaced by a torsion constant, J, which, for many practical beam sections is determined empirically (Section 11.5).

Hinges

In some cases it is convenient to impose a virtual rotation, θ_v, at some point in a structural member where, say, the actual bending moment is M_A. The internal virtual work done by M_A is then $M_A\theta_v$, (see Eq. (15.3)); physically this situation is equivalent to inserting a hinge at the point.

Sign of internal virtual work

So far we have derived expressions for internal work without considering whether it is positive or negative in relation to external virtual work.

Suppose that the structural member, AB, in Fig. 15.9(a) is, say, a member of a truss and that it is in equilibrium under the action of two externally applied axial tensile loads, P; clearly the internal axial, that is normal, force at any section of the member is P. Suppose now that the member is given a virtual extension, δ_v, such that B moves to B'. Then the virtual work done by the applied load, P, is positive since the displacement, δ_v, is in the same direction as its line of action. However, the virtual work done by the internal force, N ($=P$), is negative since the displacement of B is in the opposite direction to its line of action; in other words work is done *on* the member. Thus, from Eq. (15.8), we see that in this case

$$W_e = W_i \qquad (15.23)$$

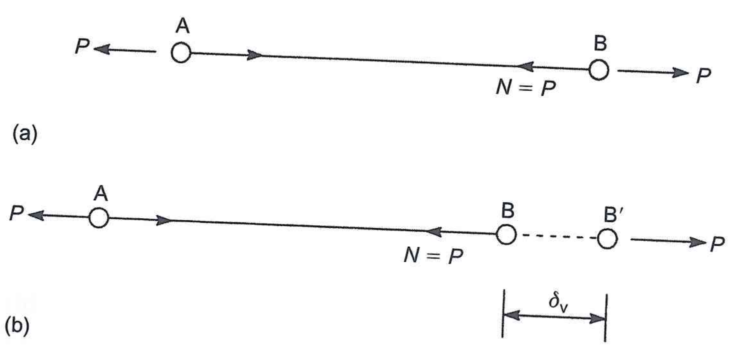


FIGURE 15.9

Sign of the internal virtual work

Equation (15.23) would apply if the virtual displacement had been a contraction and not an extension, in which case the signs of the external and internal virtual work in Eq. (15.8) would have been reversed. Clearly the above applies equally if P is a compressive load. The above arguments may be extended to structural members subjected to shear, bending and torsional loads, so that Eq. (15.23) is generally applicable.

Virtual work due to external force systems

So far in our discussion we have only considered the virtual work produced by externally applied concentrated loads. For completeness we must also consider the virtual work produced by moments, torques and distributed loads.

In Fig. 15.10 a structural member carries a distributed load, $w(x)$, and at a particular point a concentrated load, W, a moment, M, and a torque, T. Suppose that at the point a virtual displacement is imposed that has translational components, $\Delta_{v,y}$ and $\Delta_{v,x}$, parallel to the y and x axes, respectively, and rotational components, θ_v, and ϕ_v, in the yx and zy planes, respectively.

If we consider a small element, δx, of the member at the point, the distributed load may be regarded as constant over the length δx and acting, in effect, as a concentrated load $w(x)\,\delta x$. Thus the virtual work, w_e, done by the complete external force system is given by

$$w_e = W\Delta_{v,y} + P\Delta_{v,x} + M\theta_v + T\phi_v + \int_L w(x)\Delta_{v,y}\, dx$$

For a structure comprising a number of load positions, the total external virtual work done is then

$$W_e = \sum \left[W\Delta_{v,y} + P\Delta_{v,x} + M\theta_v + T\phi_v + \int_L w(x)\Delta_{v,y}\, dx \right] \qquad (15.24)$$

In Eq. (15.24) there need not be a complete set of external loads applied at every loading point so, in fact, the summation is for the appropriate number of loads. Further, the virtual displacements in the above are related to forces and moments applied in a vertical plane. We could, of course, have forces and moments and components of the virtual displacement in a horizontal plane, in which case Eq. (15.24) would be extended to include their contribution.

The internal virtual work equivalent of Eq. (15.24) for a linear system is, from Eqs (15.12), (15.17), (15.21) and (15.22)

$$W_i = \sum \left[\int_L \frac{N_A N_v}{EA}\, dx + \beta \int_L \frac{S_A S_v}{GA}\, dx + \int_L \frac{M_A M_v}{EI}\, dx + \int_L \frac{T_A T_v}{GJ}\, dx + M_A\theta_v \right] \qquad (15.25)$$

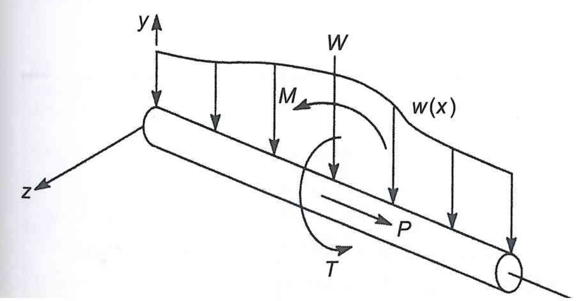


FIGURE 15.10

in which the last term on the right-hand side is the virtual work produced by an actual internal moment at a hinge (see above). Note that the summation in Eq. (15.25) is taken over all the *members* of the structure.

Use of virtual force systems

So far, in all the structural systems we have considered, virtual work has been produced by actual forces moving through imposed virtual displacements. However, the actual forces are not related to the virtual displacements in any way since, as we have seen, the magnitudes and directions of the actual forces are unchanged by the virtual displacements so long as the displacements are small. Thus the principle of virtual work applies for *any* set of forces in equilibrium and *any* set of displacements. Equally, therefore, we could specify that the forces are a set of virtual forces *in equilibrium* and that the displacements are actual displacements. Therefore, instead of relating actual external and internal force systems through virtual displacements, we can relate actual external and internal displacements through virtual forces.

If we apply a virtual force system to a deformable body it will induce an internal virtual force system which will move through the actual displacements; thus, internal virtual work will be produced. In this case, for example, Eq. (15.10) becomes

$$w_{i,N} = \int_L N_v \varepsilon_A \, dx$$

in which N_v is the internal virtual normal force and ε_A is the actual strain. Then, for a linear system, in which the actual internal normal force is N_A, $\varepsilon_A = N_A/EA$, so that for a structure comprising a number of members the total internal virtual work due to a virtual normal force is

$$W_{i,N} = \sum \int_L \frac{N_v N_A}{EA} \, dx$$

which is identical to Eq. (15.12). Equations (15.17), (15.21) and (15.22) may be shown to apply to virtual force systems in a similar manner.

Applications of the principle of virtual work

We have now seen that the principle of virtual work may be used either in the form of imposed virtual displacements or in the form of imposed virtual forces. Generally the former approach, as we saw in Exs 15.1–15.4, is used to determine forces, while the latter is used to obtain displacements.

For statically determinate structures the use of virtual displacements to determine force systems is a relatively trivial use of the principle although problems of this type provide a useful illustration of the method. The real power of this approach lies in its application to the solution of statically indeterminate structures, as we shall see in Chapter 16. However, the use of virtual forces is particularly useful in determining actual displacements of structures. We shall illustrate both approaches by examples.

EXAMPLE 15.5

Determine the bending moment at the point B in the simply supported beam ABC shown in Fig. 15.11(a).

We determined the support reactions for this particular beam in Ex. 15.4. In this example, however, we are interested in the actual internal moment, M_B, at the point of application of the load. We must therefore impose a virtual displacement which will relate the internal moment at B to the applied load and which will exclude other unknown external forces such as the support reactions and unknown internal force systems such

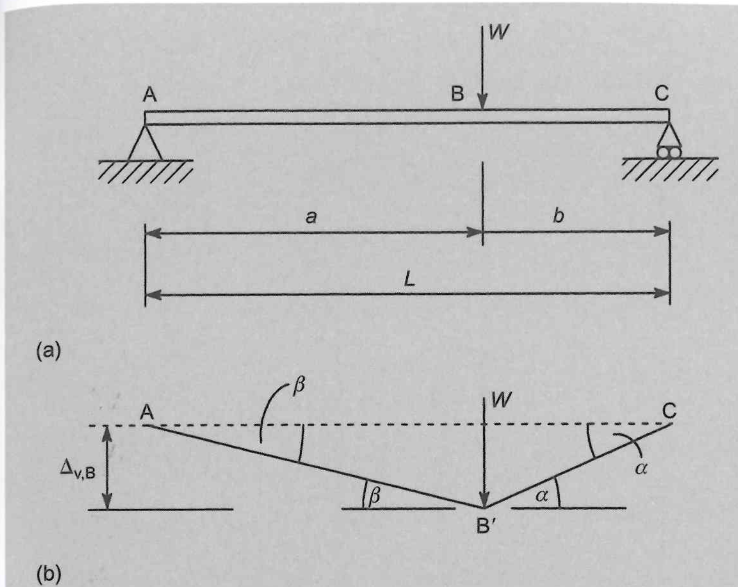


FIGURE 15.11

Determination of bending moment at a point in the beam of Ex. 15.5 using virtual work.

beam. Therefore, if we imagine that the beam is hinged at B and that the lengths AB and BC are rigid, a virtual displacement, $\Delta_{v,B}$, at B will result in the displaced shape shown in Fig. 15.11(b).

Note that the support reactions at A and C do no work and that the internal moments in AB and BC do no work because AB and BC are rigid links. From Fig. 15.11(b)

$$\Delta_{v,B} = a\beta = b\alpha \qquad \text{(i)}$$

Hence

$$\alpha = \frac{1a}{b}\beta$$

and the angle of rotation of BC relative to AB is then

$$\theta_B = \beta + \alpha = \beta\left(1 + \frac{a}{b}\right) = \frac{L}{b}\beta \qquad \text{(ii)}$$

Now equating the external virtual work done by W to the internal virtual work done by M_B (see Eq. (15.23)) we have

$$W\Delta_{v,B} = M_B\theta_B \qquad \text{(iii)}$$

Substituting in Eq. (iii) for $\Delta_{v,B}$ from Eq. (i) and for θ_B from Eq. (ii) we have

$$Wa\beta = M_B\frac{L}{b}\beta$$

which gives

$$M_B = \frac{Wab}{L}$$

which is the result we would have obtained by calculating the moment of R_C (= Wa/L from Ex.

EXAMPLE 15.6

Determine the force in the member AB in the truss shown in Fig. 15.12(a).

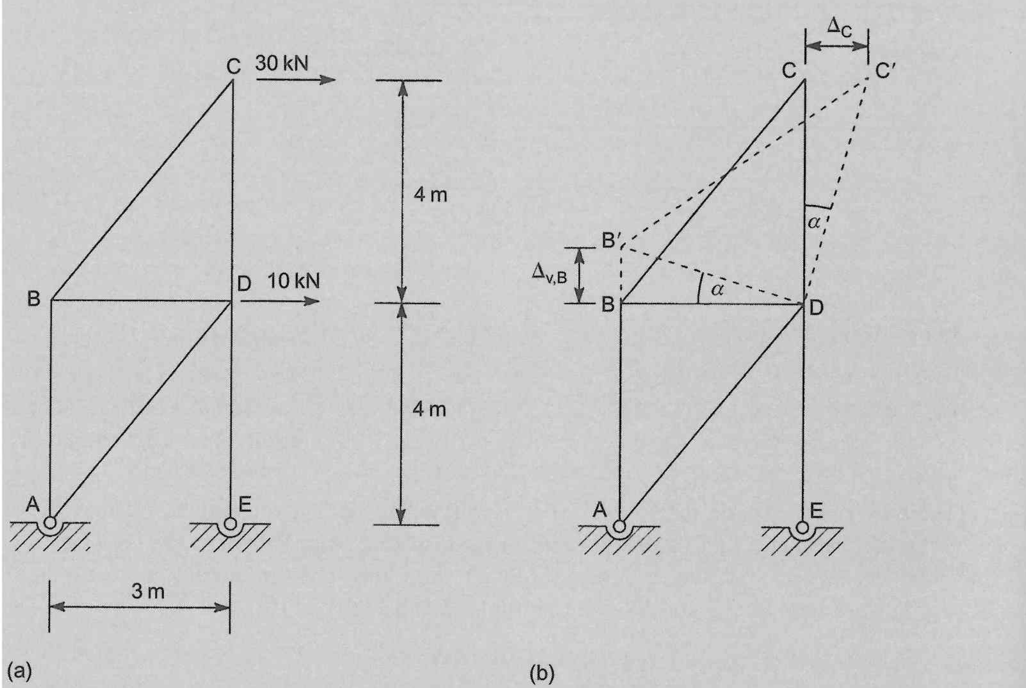


FIGURE 15.12

Determination of the internal force in a member of a truss using virtual work.

We are required to calculate the force in the member AB, so that again we need to relate this internal force to the externally applied loads without involving the internal forces in the remaining members of the truss. We therefore impose a virtual extension, $\Delta_{v,B}$, at B in the member AB, such that B moves to B'. If we assume that the remaining members are rigid, the forces in them will do no work. Further, the triangle BCD will rotate as a rigid body about D to B'C'D as shown in Fig. 15.12(b). The horizontal displacement of C, Δ_C, is then given by

$$\Delta_C = 4\alpha$$

while

$$\Delta_{v,B} = 3\alpha$$

Hence

$$\Delta_C = \frac{4\Delta_{v,B}}{3} \qquad \text{(i)}$$

Equating the external virtual work done by the 30 kN load to the internal virtual work done by the force, F_{BA}, in the member, AB, we have (see Eq. (15.23) and Fig. 15.9)

$$30\Delta_C = F_{BA}\Delta_{v,B} \qquad \text{(ii)}$$

Substituting for Δ_C from Eq. (i) in Eq. (ii),

$$30 \times \frac{4}{3}\Delta_{v,B} = F_{BA}\Delta_{v,B}$$

Whence

$$F_{BA} = +40 \text{ kN} \quad (\text{i.e. } F_{BA} \text{ is tensile})$$

In the above we are, in effect, assigning a positive (i.e. tensile) sign to F_{BA} by imposing a virtual extension on the member AB.

The actual sign of F_{BA} is then governed by the sign of the external virtual work. Thus, if the 30 kN load had been in the opposite direction to Δ_C the external work done would have been negative, so that F_{BA} would be negative and therefore compressive. This situation can be verified by inspection. Alternatively, for the loading as shown in Fig. 15.12(a), a contraction in AB would have implied that F_{BA} was compressive. In this case DC would have rotated in an anticlockwise sense, Δ_C would have been in the opposite direction to the 30 kN load so that the external virtual work done would be negative, resulting in a negative value for the compressive force F_{BA}; F_{BA} would therefore be tensile as before. Note also that the 10 kN load at D does no work since D remains undisplaced.

We shall now consider problems involving the use of virtual forces. Generally we shall require the displacement of a particular point in a structure, so that if we apply a virtual force to the structure at the point and in the direction of the required displacement the external virtual work done will be the product of the virtual force and the actual displacement, which may then be equated to the internal virtual work produced by the internal virtual force system moving through actual displacements. Since the choice of the virtual force is arbitrary, we may give it any convenient value; the simplest type of virtual force is therefore a unit load and the method then becomes the *unit load method*.

EXAMPLE 15.7

Determine the vertical deflection of the free end of the cantilever beam shown in Fig. 15.13(a).

Let us suppose that the actual deflection of the cantilever at B produced by the uniformly distributed load is v_B and that a vertically downward virtual unit load was applied at B before the actual deflection took place. The external virtual work done by the unit load is, from Fig. 15.13(b), $1v_B$. The deflection, v_B, is assumed to be caused by bending only, i.e. we are ignoring any deflections due to shear. The internal virtual work is given by Eq. (15.21) which, since only one member is involved, becomes

$$W_{i,M} = \int_0^L \frac{M_A M_v}{EI}\,dx \qquad \text{(i)}$$

The virtual moments, M_v, are produced by a unit load so that we shall replace M_v by M_1. Then

$$W_{i,M} \int_0^L \frac{M_A M_1}{EI}\,dx \qquad \text{(ii)}$$

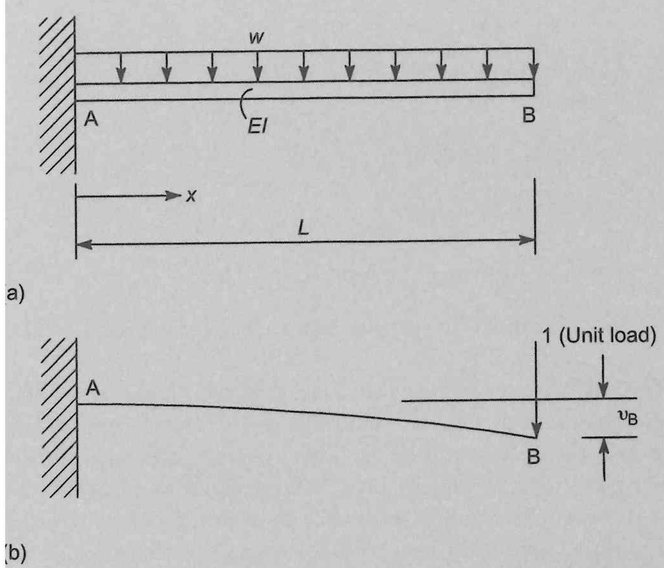


FIGURE 15.13

Deflection of the free end of a cantilever beam using the unit load method.

At any section of the beam a distance x from the built-in end

$$M_A = -\frac{w}{2}(L-x)^2 \quad M_1 = -1(L-x)$$

Substituting for M_A and M_1 in Eq. (ii) and equating the external virtual work done by the unit ad to the internal virtual work we have

$$1v_B = \int_0^L \frac{w}{2EI}(L-x)^3 dx$$

hich gives

$$v_B = -\frac{w}{2EI}\left[\frac{1}{4}(L-x)^4\right]_0^L$$

that

$$v_B = \frac{wL^4}{8EI} \quad \text{(as in Ex. 13.2)}$$

Note that v_B is in fact negative but the positive sign here indicates that it is in the same direction the unit load.

XAMPLE 15.8

etermine the rotation, i.e. the slope, of the beam ABC shown in Fig. 15.14(a) at A.

The actual rotation of the beam at A produced by the actual concentrated load, W, is θ_A. Let us ıppose that a virtual unit moment is applied at A before the actual rotation takes place, as shown Fig. 15.14(b). The virtual unit moment induces virtual support reactions of $R_{v,A}$ $(=1/L)$ acting

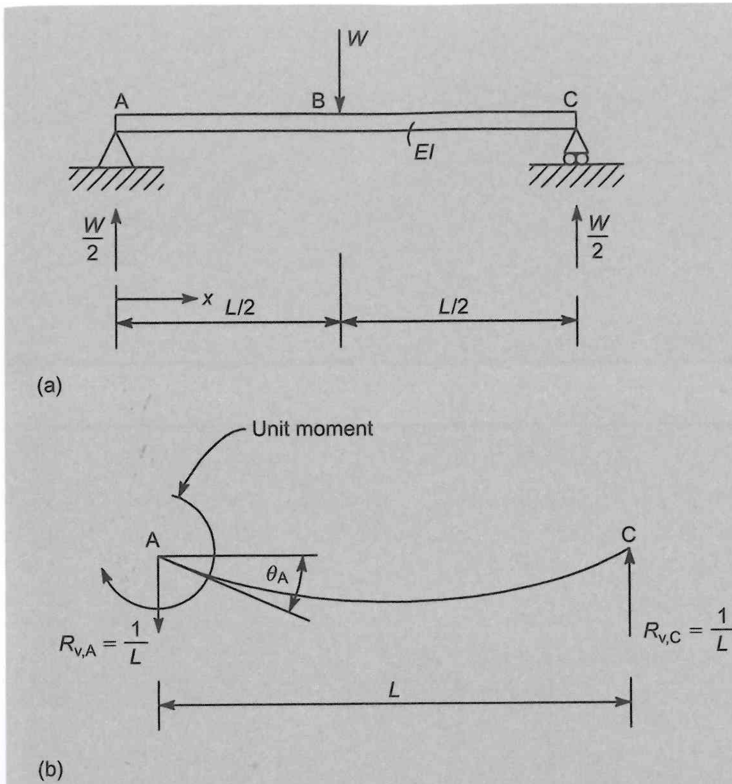


FIGURE 15.14

Determination of the rotation of a simply supported beam at a support using the unit load method.

$$M_A = +\frac{W}{2}x \quad 0 \leq x \leq L/2$$

$$M_A = +\frac{W}{2}(L-x) \quad L/2 \leq x \leq L$$

The internal virtual bending moment is

$$M_v = 1 - \frac{1}{L}x \quad 0 \leq x \leq L$$

The external virtual work done is $1\theta_A$ (the virtual support reactions do no work as there is no vertical displacement of the beam at the supports) and the internal virtual work done is given by Eq. (15.21). Hence

$$1\theta_A = \frac{1}{EI}\left[\int_0^{L/2} \frac{W}{2}x\left(1-\frac{x}{L}\right) dx + \int_{L/2}^{L} \frac{W}{2}(L-x)\left(1-\frac{x}{L}\right) dx\right] \quad \text{(i)}$$

Simplifying Eq. (i) we have

$$\theta_A = \frac{W}{2EIL}\left[\int_0^{L/2} (Lx - x^2)\, dx + \int_{L/2}^{L} (L-x)^2\, dx\right] \quad \text{(ii)}$$

Hence

$$\theta_A = \frac{W}{2EIL}\left\{\left[L\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{L/2} - \frac{1}{3}\left[(L-x)^3\right]_{L/2}^{L}\right\}$$

from which

$$\theta_A = \frac{WL^2}{16EI}$$

which is the result that may be obtained from Eq. (iii) of Ex. 13.5.

EXAMPLE 15.9

Calculate the vertical deflection of the joint B and the horizontal movement of the support D in the truss shown in Fig. 15.15(a). The cross-sectional area of each member is 1800 mm^2 and Young's modulus, E, for the material of the members is 200 000 N/mm^2.

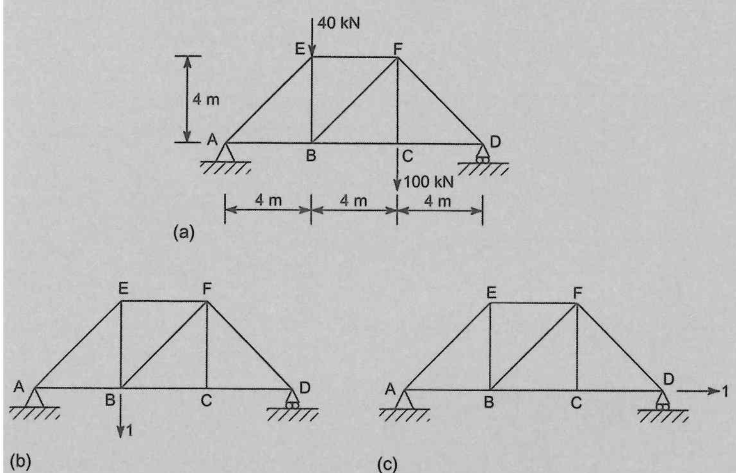


FIGURE 15.15

Deflection of a truss using the unit load method.

The virtual force systems, i.e. unit loads, required to determine the vertical deflection of B and the horizontal deflection of D are shown in Fig. 15.15(b) and (c), respectively. Therefore, if the actual vertical deflection at B is $\delta_{B,v}$ and the horizontal deflection at D is $\delta_{D,h}$ the external virtual work done by the unit loads is $1\delta_{B,v}$ and $1\delta_{D,h}$, respectively. The internal actual and virtual force systems comprise axial forces in all the members. These axial forces are constant along the length of each member so that for a truss comprising n members, Eq. (15.12) reduces to

$$W_{i,N} = \sum_{j=1}^{n} \frac{F_{A,j}F_{v,j}L_j}{E_jA_j} \qquad \text{(i)}$$

in which $F_{A,j}$ and $F_{v,j}$ are the actual and virtual forces in the jth member which has a length L_j, an area of cross section A_j and a Young's modulus E_j.

Since the forces $F_{v,j}$ are due to a unit load, we shall write Eq. (i) in the form

$$W_{i,N} = \sum_{j=1}^{n} \frac{F_{A,j}F_{1,j}L_j}{E_jA_j} \qquad \text{(ii)}$$

Also, in this particular example, the area of cross section, A, and Young's modulus, E, are the same for all members so that it is sufficient to calculate $\sum_{j=1}^{n} F_{A,j}F_{1,j}L_j$ and then divide by EA to obtain $W_{i,N}$.

The forces in the members, whether actual or virtual, may be calculated by the method of joints (Section 4.6). Note that the support reactions corresponding to the three sets of applied loads (one actual and two virtual) must be calculated before the internal force systems can be determined. However, in Fig. 15.15(c), it is clear from inspection that $F_{1,AB} = F_{1,BC} = F_{1,CD} = +1$ while the forces in all other members are zero. The calculations are presented in Table 15.1; note that positive signs indicate tension and negative signs compression.

Table 15.1

Member	*L* (m)	F_A(kN)	$F_{1,B}$	$F_{1,D}$	$F_AF_{1,B}L$(kNm)	$F_AF_{1,D}L$ (kNm)
AE	5.7	−84.9	−0.94	0	+451.4	0
AB	4.0	+60.0	+0.67	+1.0	+160.8	+240.0
EF	4.0	−60.0	−0.67	0	+160.8	0
EB	4.0	+20.0	+0.67	0	+53.6	0
BF	5.7	−28.3	+0.47	0	−75.2	0
BC	4.0	+80.0	+0.33	+1.0	+105.6	+320.0
CD	4.0	+80.0	+0.33	+1.0	+105.6	+320.0
CF	4.0	+100.0	0	0	0	0
DF	5.7	−113.1	−0.47	0	+301.0	0
					$\sum = +1263.6$	$\sum = +880.0$

Thus equating internal and external virtual work done (Eq. (15.23)) we have

$$1\delta_{B,v} = \frac{1263.6 \times 10^6}{200\,000 \times 1800}$$

whence

$$\delta_{B,v} = 3.51 \text{ mm}$$

and

$$1\delta_{D,h} = \frac{880 \times 10^6}{200\,000 \times 1800}$$

which gives

$$\delta_{D,h} = 2.44 \text{ mm}$$

Both deflections are positive which indicates that the deflections are in the directions of the applied unit loads. Note that in the above it is unnecessary to specify units for the unit load since the unit load appears, in effect, on both sides of the virtual work equation (the internal F_1 forces are directly proportional to the unit load).

EXAMPLE 15.10

Determine the horizontal and vertical components of the deflection of the point C in the frame shown in Fig. 15.16(a); consider the effects of bending only.

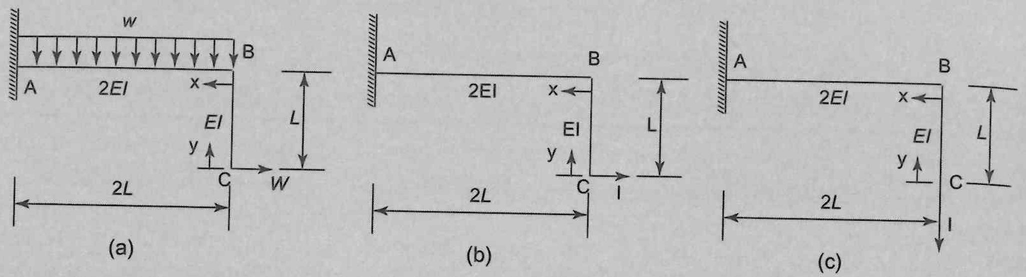


FIGURE 15.16

Frame of Ex. 15.10.

The components of the deflection of C may be found by applying horizontal and vertical unit loads in turn at C as shown in Figs. 15.16(b) and (c). The internal virtual work done by this virtual force system, that is unit loads acting through real displacements, is given by Eq. (15.20) in which, for AB

$$M_A = WL - (wx^2/2), \quad M_v = 1L \text{ (horiz.)}, \quad M_v = -1x \text{ (vert.)}$$

and for BC

$$M_A = Wy, \quad M_v = 1y \text{ (horiz.)}, \quad M_v = 0 \text{ (vert.)}$$

Considering the horizontal component of deflection first, the total internal work done is

$$W_t = \int_0^L (wy^2/EI)\, dy + \int_0^{2L} [WL - (wx^2/2)](L/2EI)\, dx$$

that is

$$W_t = (W/EI)\,[y^3/3]_0^L + (L/2EI)[WLx - (wx^3/6)]_0^{2L}$$

which gives

$$W_t = 2L^3\,(2W - wL)/3EI \qquad \text{(i)}$$

The virtual external work done by the unit load is $1\delta_{C,h}$ where $\delta_{C,h}$ is the horizontal component of the deflection of C. Equating this to the total internal virtual work given by Eq. (i) gives

$$\delta_{C,h} = 2L^3\,(2W - wL)/3EI \qquad \text{(ii)}$$

Now considering the vertical component of the deflection

$$W_t = (1/2EI)\int_0^{2L} [WL - (wx^2/2)]\,(-x)\, dx$$

Note that for BC, $M_v = 0$. Integrating this expression and substituting the limits

$$W_t = L^3\,(-W + wL)/EI \qquad \text{(iii)}$$

The external work done is $1\delta_{C,v}$ where $\delta_{C,v}$ is the vertical component of the deflection of C. Equating the internal and external virtual work gives

$$\delta_{C,v} = L^3\,(-W + wL)/EI \qquad \text{(iv)}$$

Note that the components of deflection can be either positive or negative depending on the relative magnitudes of W and w. A positive value indicates a deflection in the direction of the applied unit load while a negative value indicates a deflection in the opposite direction to the applied unit load.

EXAMPLE 15.11

The cantilever beam AB shown in plan in Fig. 15.17 takes the form of a quadrant of a circle and is positioned in a horizontal plane. If the beam supports a vertically downward load, W, at its free end B and its bending and torsional stiffnesses are EI and GJ respectively, calculate the vertical component of the deflection of B.

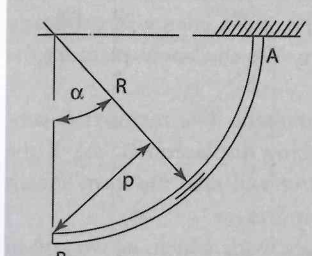


FIGURE 15.17

Cantilever beam of Ex. 15.11.

To determine the vertical displacement we apply a virtual unit load at B vertically downwards (i.e into the plane of the paper).

At a section of the beam where the radius at the section makes an angle, α, with the radius through B

$$M_A = Wp = WR\sin\alpha, \quad M_v = 1R\sin\alpha$$
$$T_A = W(R - R\cos\alpha), \quad T_v = 1(R - R\cos\alpha)$$

The total internal virtual work done is given by the summation of Eqs. (15.20) and (15.22), that is

$$W_t = (1/EI)\int_0^{\pi/2} WR^2\sin^2\alpha\, R\, d\alpha + (1/GJ)\int_0^{\pi/2} WR^2(1-\cos\alpha)^2\, R\, d\alpha \qquad \text{(i)}$$

Integrating Eq. (i) and substituting the limits gives

$$W_t = WR^3\{(\pi/4EI) + (1/GJ)[(3\pi/4) - 2]\} \qquad \text{(ii)}$$

The external work done by the unit load is $1\delta_B$ so that equating with Eq. (ii) gives

$$\delta_B = WR^3\{(\pi/4EI) + (1/GJ)[(3\pi/4) - 2]\}$$

Examples 15.5–15.11 illustrate the application of the principle of virtual work to the solution of problems involving statically determinate linearly elastic structures. We shall now examine the alternative energy methods but we shall return to the use of virtual work in Chapter 16 when we consider stat-

5.3 Energy methods

lthough it is generally accepted that energy methods are not as powerful as the principle of virtual ork in that they are limited to elastic analysis, they possibly find their greatest use in providing rapid pproximate solutions of problems for which exact solutions do not exist. Also, many statically indeterinate structures may be conveniently analysed using energy methods while, in addition, they are capale of providing comparatively simple solutions for deflection problems which are not readily solved by nore elementary means.

Energy methods involve the use of either the *total complementary energy* or the *total potential energy* TPE) of a structural system. Either method may be employed to solve a particular problem, although s a general rule displacements are more easily found using complementary energy while forces are nore easily found using potential energy.

train energy and complementary energy

n Section 7.10 we investigated strain energy in a linearly elastic member subjected to an axial load. ubsequently in Sections 9.4, 10.3 and 11.2 we derived expressions for the strain energy in a linearly lastic member subjected to bending, shear and torsional loads, respectively. We shall now examine the nore general case of a member that is not linearly elastic.

Figure 15.18(a) shows the jth member of a structure comprising n members. The member is subected to a gradually increasing load, P_j, which produces a gradually increasing displacement, Δ_j. If the nember possesses non-linear elastic characteristics, the load–deflection curve will take the form shown n Fig. 15.18(b). Let us suppose that the final values of P_j and Δ_j are $P_{j,F}$ and $\Delta_{j,F}$.

As the member extends (or contracts if P_j is a compressive load) P_j does work which, as we saw in ection 7.10, is stored in the member as strain energy. The work done by P_j as the member extends by small amount $\delta\Delta_j$ is given by

$$\delta W_j = P_j \delta \Delta_j$$

Therefore the total work done by P_j, and therefore the strain energy stored in the member, as P_j ncreases from zero to $P_{j,F}$ is given by

$$u_j = \int_0^{\Delta_{j,F}} P_j \mathrm{d}\Delta_j \qquad (15.26)$$

vhich is clearly the area OBD under the load–deflection curve in Fig. 15.18(b). Similarly the area)AB, which we shall denote by c_j, above the load–deflection curve is given by

$$c_j = \int_0^{P_{j,F}} \Delta_j \mathrm{d}P_j \qquad (15.27)$$

It may be seen from Fig. 15.18(b) that the area OABD represents the work done by a constant force $P_{j,F}$ moving through the displacement $\Delta_{j,F}$. Thus from Eqs (15.26) and (15.27)

$$u_j + c_j = P_{j,F}\Delta_{j,F} \qquad (15.28)$$

It follows that since u_j has the dimensions of work, c_j also has the dimensions of work but otherwise c_j has no physical meaning. It can, however, be regarded as the complement of the work done by P_j in

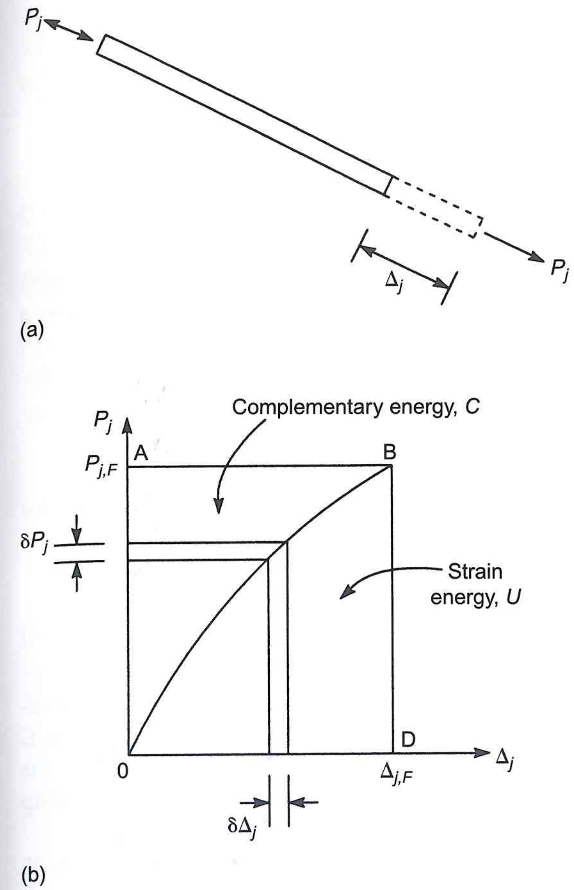


FIGURE 15.18

Load–deflection curve for a non-linearly elastic member.

The total strain energy, U, of the structure is the sum of the individual strain energies of the members. Thus

$$U = \sum_{j=1}^{n} u_j$$

which becomes, when substituting for u_j from Eq. (15.26)

$$U = \sum_{j=1}^{n} \int_0^{\Delta_{j,F}} P_j \mathrm{d}\Delta_j \qquad (15.29)$$

Similarly, the total complementary energy, C, of the structure is given by

$$C = \sum_{j=1}^{n} c_j$$

whence, from Eq. (15.27)

$$C = \sum_{j=1}^{n} \int_0^{P_{j,F}} \Delta_j \mathrm{d}P_j \qquad (15.30)$$

Equation (15.29) may be written in expanded form as

$$U = \int_0^{\Delta_{1,F}} P_1\,d\Delta_1 + \int_0^{\Delta_{2,F}} P_2\,d\Delta_2 + \ldots + \int_0^{\Delta_{j,F}} P_j\,d\Delta_j + \ldots + \int_0^{\Delta_{n,F}} P_n\,d\Delta_n \qquad (15.31)$$

Partially differentiating Eq. (15.31) with respect to a particular displacement, say Δ_j, gives

$$\frac{\partial U}{\partial \Delta_j} = P_j \qquad (15.32)$$

Equation (15.32) states that the partial derivative of the strain energy in an elastic structure with respect to a displacement Δ_j is equal to the corresponding force P_j; clearly U must be expressed as a function of the displacements. This equation is generally known as *Castigliano's first theorem (Part I)* after the Italian engineer who derived and published it in 1879. One of its primary uses is in the analysis of non-linearly elastic structures, which is outside the scope of this book.

Now writing Eq. (15.30) in expanded form we have

$$C = \int_0^{P_{1,F}} \Delta_1\,dP_1 + \int_0^{\Delta P_{2,F}} \Delta_2\,dP_2 + \ldots + \int_0^{P_{j,F}} \Delta_j\,dP_j + \ldots + \int_0^{P_{n,F}} \Delta_n\,dP_n \qquad (15.33)$$

The partial derivative of Eq. (15.33) with respect to one of the loads, say P_j, is then

$$\frac{\partial C}{\partial P_j} = \Delta_j \qquad (15.34)$$

Equation (15.34) states that the partial derivative of the complementary energy of an elastic structure with respect to an applied load, P_j, gives the displacement of that load in its own line of action; C in this case is expressed as a function of the loads. Equation (15.34) is sometimes called the *Crotti–Engesser theorem* after the two engineers, one Italian, one German, who derived the relationship independently, Crotti in 1879 and Engesser in 1889.

Now consider the situation that arises when the load–deflection curve is linear, as shown in Fig. 15.19. In this case the areas OBD and OAB are equal so that the strain and complementary energies are equal. Thus we may replace the complementary energy, C, in Eq. (15.34) by the strain energy, U. Hence

$$\frac{\partial U}{\partial P_j} = \Delta_j \qquad (15.35)$$

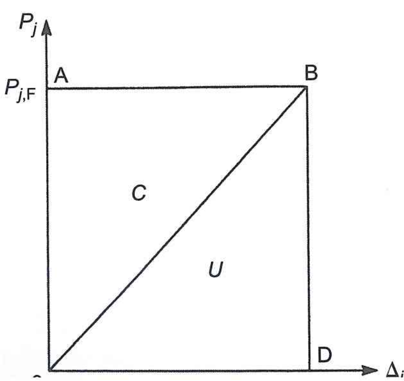


FIGURE 15.19

Equation (15.35) states that, for a linearly elastic structure, the partial derivative of the strain energy of a structure with respect to a load gives the displacement of the load in its own line of action. This is generally known as *Castigliano's first theorem (Part II)*. Its direct use is limited in that it enables the displacement at a particular point in a structure to be determined *only* if there is a load applied at the point and *only* in the direction of the load. It could not therefore be used to solve for the required displacements at B and D in the truss in Ex. 15.9.

The principle of the stationary value of the total complementary energy

Suppose that an elastic structure comprising n members is in equilibrium under the action of a number of forces, $P_1, P_2, \ldots, P_k, \ldots, P_r$ which produce corresponding actual displacements, $\Delta_1, \Delta_2, \ldots, \Delta_k, \ldots, \Delta_r$, and actual internal forces, $F_1, F_2, \ldots, F_j, \ldots, F_n$. Now let us suppose that a system of elemental virtual forces, $\delta P_1, \delta P_2, \ldots, \delta P_k, \ldots, \delta P_r$, are imposed on the structure and act through the actual displacements. The external virtual work, δW_e, done by these elemental virtual forces is, from Section 15.2,

$$\delta W_e = \delta P_1\,\Delta_1 + \delta P_2\,\Delta_2 + \ldots + \delta P_k\,\Delta_k + \ldots + \delta P_r\Delta_r$$

or

$$\delta W_e = \sum_{k=1}^{r} \Delta_k \delta P_k \qquad (15.36)$$

At the same time the elemental external virtual forces are in equilibrium with an elemental internal virtual force system, $\delta F_1, \delta F_2, \ldots, \delta F_j, \ldots, \delta F_n$, which moves through actual internal deformations, $\delta_1, \delta_2, \ldots, \delta_j, \ldots, \delta_n$. Hence the internal elemental virtual work done is

$$\delta W_i = \sum_{j=1}^{n} \delta_j \delta F_j \qquad (15.37)$$

From Eq. (15.23)

$$\sum_{k=1}^{r} \Delta_k \delta P_k = \sum_{j=1}^{n} \delta_j \delta F_j$$

so that

$$\sum_{j=1}^{n} \delta_j \delta F_j - \sum_{k=1}^{r} \Delta_k \delta P_k = 0 \qquad (15.38)$$

Equation (15.38) may be written as

$$\delta\left(\sum_{j=1}^{n} \int_0^{F_j} \delta_j dF_j - \sum_{k=1}^{r} \Delta_k P_k\right) = 0 \qquad (15.39)$$

From Eq. (15.30) we see that the first term in Eq. (15.39) represents the complementary energy, C_i, of the actual internal force system, while the second term represents the complementary energy, C_e, of the external force system. C_i and C_e are opposite in sign since C_e is the complement of the work done *by* the external force system while C_i is the complement of the work done *on* the structure. Rewriting Eq. (15.39), we have

$$\delta(C_i + C_e) = 0 \qquad (15.40)$$

In Eq. (15.39) the displacements, Δ_k, and the deformations, δ_j, are the actual displacements and deformations of the elastic structure. They therefore obey the condition of compatibility of displacement so that Eqs (15.39) and (15.40) are equations of geometrical compatibility. Also Eq. (15.40) establishes the *principle of the stationary value of the total complementary energy* which may be stated as:

For an elastic body in equilibrium under the action of applied forces the true internal forces (or stresses) and reactions are those for which the total complementary energy has a stationary value.

In other words the true internal forces (or stresses) and reactions are those that satisfy the condition of compatibility of displacement. This property of the total complementary energy of an elastic structure is particularly useful in the solution of statically indeterminate structures in which an infinite number of stress distributions and reactive forces may be found to satisfy the requirements of equilibrium so that, as we have already seen, equilibrium conditions are insufficient for a solution.

We shall examine the application of the principle in the solution of statically indeterminate structures in Chapter 16. Meanwhile we shall illustrate its application to the calculation of displacements in statically determinate structures.

EXAMPLE 15.12

The calculation of deflections in a truss.

Suppose that we wish to calculate the deflection, Δ_2, in the direction of the load, P_2, and at the joint at which P_2 is applied in a truss comprising n members and carrying a system of loads P_1, $P_2, \ldots, P_k, \ldots, P_r$ as shown in Fig. 15.20. From Eq. (15.39) the total complementary energy, C, of the truss is given by

$$C = \sum_{j=1}^{n} \int_0^{F_j} \delta_j \mathrm{d}F_j - \sum_{k=1}^{r} \Delta_k P_k \qquad \text{(i)}$$

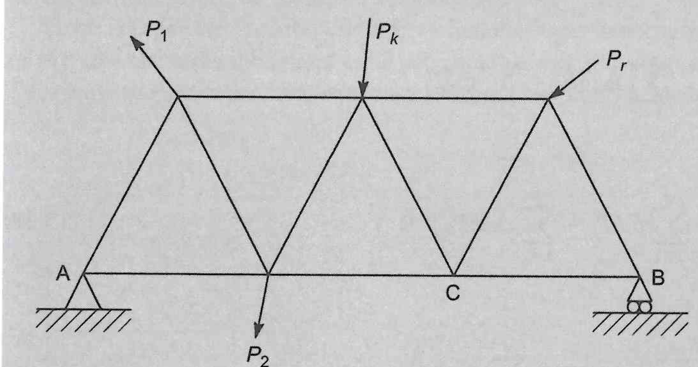


FIGURE 15.20

Deflection of a truss using complementary energy.

From the principle of the stationary value of the total complementary energy with respect to the load P_2, we have

$$\frac{\partial C}{\partial P_2} = \sum_{j=1}^{n} \delta_j \frac{\partial F_j}{\partial P_2} - \Delta_2 = 0 \qquad \text{(ii)}$$

from which

Note that the partial derivatives with respect to P_2 of the fixed loads, $P_1, P_3, \ldots, P_k, \ldots, P_r$ vanish.

To complete the solution we require the load–displacement characteristics of the structure. For a non-linear system in which, say,

$$F_j = b(\delta_j)^c$$

where b and c are known, Eq. (iii) becomes

$$\Delta_2 = \sum_{j=1}^{n} \left(\frac{F_j}{b}\right)^{1/c} \frac{\partial F_j}{\partial P_2} \qquad \text{(iv)}$$

In Eq. (iv) F_j may be obtained from basic equilibrium conditions, e.g. the method of joints, and expressed in terms of P_2; hence $\partial F_j/\partial P_2$ is found. The actual value of P_2 is then substituted in the expression for F_j and the product $(F_j/b)^{1/c}\partial F_j/\partial P_2$ calculated for each member. Summation then gives Δ_2.

In the case of a linearly elastic structure δ_j is, from Sections 7.4 and 7.7, given by

$$\delta_j = \frac{F_j}{E_j A_j} L_j$$

in which E_j, A_j and L_j are Young's modulus, the area of cross section and the length of the jth member. Substituting for δ_j in Eq. (iii) we obtain

$$\Delta_2 = \sum_{j=1}^{n} \frac{F_j L_j}{E_j A_j} \frac{\partial F_j}{\partial P_2} \qquad \text{(v)}$$

Equation (v) could have been derived directly from Castigliano's first theorem (Part II) which is expressed in Eq. (15.35) since, for a linearly elastic system, the complementary and strain energies are identical; in this case the strain energy of the jth member is $F_j^2 L_j/2A_jE_j$ from Eq. (7.29). Other aspects of the solution merit discussion.

We note that the support reactions at A and B do not appear in Eq. (i). This convenient absence derives from the fact that the displacements, $\Delta_1, \Delta_2, \ldots, \Delta_k, \ldots, \Delta_r$, are the actual displacements of the truss and fulfil the conditions of geometrical compatibility and boundary restraint. The complementary energy of the reactions at A and B is therefore zero since both of their corresponding displacements are zero.

In Eq. (v) the term $\partial F_j/\partial P_2$ represents the rate of change of the actual forces in the members of the truss with P_2. This may be found, as described in the non-linear case, by calculating the forces, F_j, in the members in terms of P_2 and then differentiating these expressions with respect to P_2. Subsequently the actual value of P_2 would be substituted in the expressions for F_j and thus, using Eq. (v), Δ_2 obtained. This approach is rather clumsy. A simpler alternative would be to calculate the forces, F_j, in the members produced by the applied loads including P_2, then remove all the loads and apply P_2 only as an unknown force and recalculate the forces F_j as functions of P_2; $\partial F_j/\partial P_2$ is then obtained by differentiating these functions.

This procedure indicates a method for calculating the displacement of a point in the truss in a direction not coincident with the line of action of a load or, in fact, of a point such as C which carries no load at all. Initially the forces F_j in the members due to $P_1, P_2, \ldots, P_k, \ldots, P_r$ are calculated. These loads are then removed and a *dummy* or *fictitious* load P_C applied at the point and in the direction of the required displacement. A new set of forces, F_j, are calculated in terms of the dummy load, P_f, and thus $\partial F_j/\partial P_f$ is obtained. The required displacement, say Δ_C of C, is then given by

$$\Delta_C = \sum_{j=1}^{n} \frac{F_j L_j}{E_j A_j}\frac{\partial F_j}{\partial P_f} \tag{vi}$$

The simplification may be taken a stage further. The force F_j in a member due to the dummy load may be expressed, since the system is linearly elastic, in terms of the dummy load as

$$F_j = \frac{\partial F_j}{\partial P_f} P_f \tag{vii}$$

Suppose now that $P_f = 1$, i.e. a *unit load.* Equation (vii) then becomes

$$F_j = \frac{\partial F_j}{\partial P_f} 1$$

so that $\partial F_j/\partial P_f = F_{1,j}$, the load in the jth member due to a unit load applied at the point and in the direction of the required displacement. Thus, Eq. (vi) may be written as

$$\Delta_C = \sum_{j=1}^{n} \frac{F_j F_{1,j} L_j}{E_j A_j} \tag{viii}$$

in which a unit load has been applied at C in the direction of the required displacement. Note that Eq. (viii) is identical in form to Eq. (ii) of Ex. 15.9.

In the above we have concentrated on members subjected to axial loads. The arguments apply in cases where structural members carry bending moments that produce rotations, shear loads that cause shear deflections and torques that produce angles of twist. We shall now demonstrate the application of the method to structures subjected to other than axial loads.

EXAMPLE 15.13

Calculate the deflection, v_B, at the free end of the cantilever beam shown in Fig. 15.21(a).

We shall assume that deflections due to shear are negligible so that v_B is entirely due to bending action in the beam. In this case the total complementary energy of the beam is, from Eq. (15.39)

$$C = \int_0^L \int_0^M d\theta\, dM - W v_B \tag{i}$$

in which M is the bending moment acting on an element, δx, of the beam; δx subtends a small angle, $\delta\theta$, at the centre of curvature of the beam. The radius of curvature of the beam at the section is R as shown in Fig. 15.21(b) where, for clarity, we represent the beam by its neutral plane. From the principle of the stationary value of the total complementary energy of the beam

$$\frac{\partial C}{\partial W} = \int_0^L \frac{\partial M}{\partial W} d\theta - v_B = 0$$

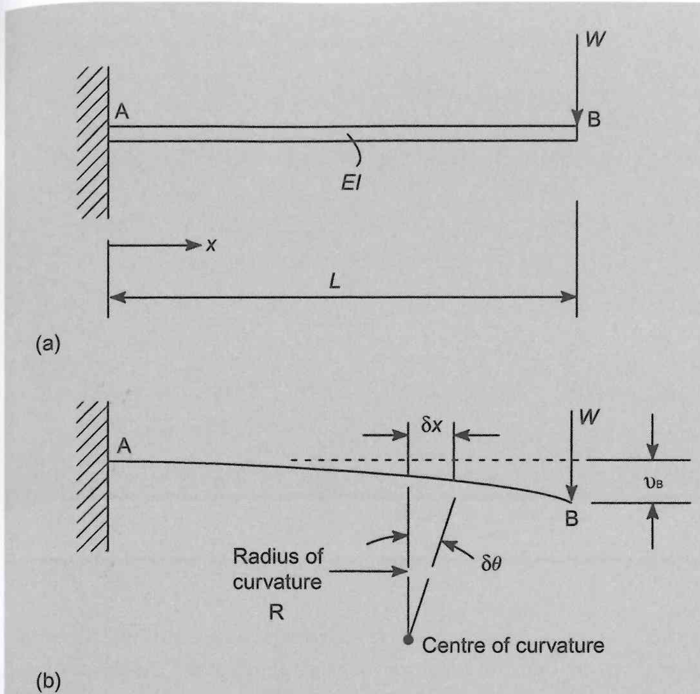


FIGURE 15.21
Deflection of a cantilever beam using complementary energy.

whence

$$v_B = \int_0^L \frac{\partial M}{\partial W} d\theta \tag{ii}$$

In Eq. (ii)

$$\delta\theta = \frac{\delta x}{R}$$

and from Eq. (9.11)

$$-\frac{1}{R} = \frac{M}{EI}$$

(here the curvature is negative since the centre of curvature is below the beam) so that

$$\delta\theta = -\frac{M}{EI}\delta x$$

Substituting in Eq. (ii) for $\delta\theta$ we have

$$v_B = -\int_0^L \frac{M}{EI}\frac{\partial M}{\partial W} dx \tag{iii}$$

From Fig. 15.21(a) we see that

$$M = -W(L - x)$$

Hence

$$\frac{\partial M}{\partial W} = -(L-x)$$

Note: Equation (iii) could have been obtained directly from Eq. (9.21) by using Castigliano's first theorem (Part II).

Equation (iii) then becomes

$$v_B = -\int_0^L \frac{W}{EI}(L-x)^2 \mathrm{d}x$$

whence

$$v_B = -\frac{WL^3}{3EI} \quad \text{(as in Ex 13.1)}$$

(Note that v_B is downwards and therefore negative according to our sign convention.)

EXAMPLE 15.14

Determine the deflection, v_B, of the free end of a cantilever beam carrying a uniformly distributed load of intensity w. The beam is represented in Fig. 15.22 by its neutral plane; the flexural rigidity of the beam is EI.

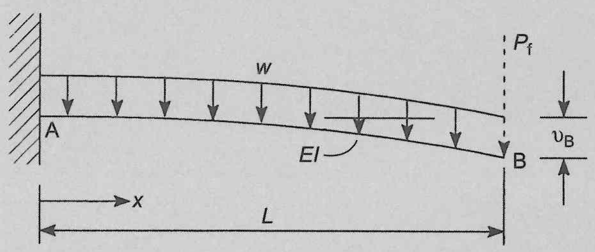


FIGURE 15.22
Deflection of a cantilever beam using the dummy load method.

For this example we use the dummy load method to determine v_B since we require the deflection at a point which does not coincide with the position of a concentrated load; thus we apply a dummy load, P_f, at B as shown. The total complementary energy, C, of the beam includes that produced by the uniformly distributed load; thus

$$C = \int_0^L \int_0^M \mathrm{d}\theta\, \mathrm{d}M - P_f v_B - \int_0^L vw\, \mathrm{d}x \qquad \text{(i)}$$

in which v is the displacement of an elemental length, δx, of the beam at any distance x from the built-in end. Then

$$\frac{\partial C}{\partial P_f} = \int_0^L \mathrm{d}\theta \frac{\partial M}{\partial P_f} - v_B = 0$$

so that

Note that in Eq. (i) v is an actual displacement and w an actual load, so that the last term disappears when C is partially differentiated with respect to P_f. As in Ex. 15.13

$$\delta\theta = -\frac{M}{EI}\delta x$$

Also

$$M = -P_f(L-x) - \frac{w}{2}(L-x)^2$$

in which P_f is imaginary and therefore disappears when we substitute for M in Eq. (ii). Then

$$\frac{\partial M}{\partial P_f} = -(L-x)$$

so that

$$v_B = -\int_0^L \frac{w}{2EI}(L-x)^2 \mathrm{d}x$$

whence

$$v_B = -\frac{wL^4}{8EI} \quad \text{(see Ex. 13.2)}$$

For a linearly elastic system the bending moment, M_f, produced by a dummy load, P_f, may be written as

$$M_f = \frac{\partial M}{\partial P_f} P_f$$

If $P_f = 1$, i.e. a *unit load*

$$M_f = \frac{\partial M}{\partial P_f} 1$$

so that $\partial M/\partial P_f = M_1$, the bending moment due to a unit load applied at the point and in the direction of the required deflection. Thus we could write an equation for deflection, such as Eq. (ii), in the form

$$v = \int_0^L \frac{M_A M_1}{EI} \mathrm{d}x \quad \text{(see Eq. (ii) of Ex.15.7)} \qquad \text{(iii)}$$

in which M_A is the actual bending moment at any section of the beam and M_1 is the bending moment at any section of the beam due to a unit load applied at the point and in the direction of the required deflection. Thus, in this example

$$M_A = -\frac{w}{2}(L-x)^2 \quad M_1 = -1(L-x)$$

so that

$$v_B = \int_0^L \frac{w}{2EI}(L-x)^3 \mathrm{d}x$$

EXAMPLE 15.15

Calculate the vertical displacements at the quarter and mid-span points B and C in the simply supported beam shown in Fig. 15.23. The flexural rigidity of the beam is EI.

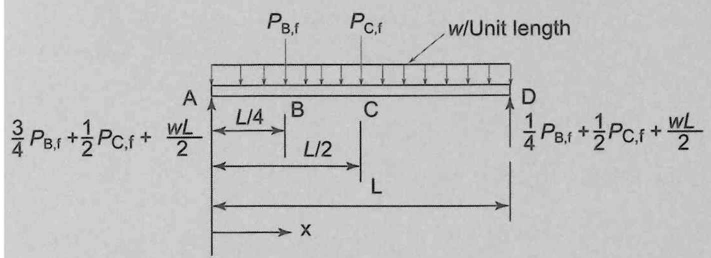


FIGURE 15.23
Beam deflection of Ex. 15.15.

We apply fictitious loads, $P_{B,f}$ and $P_{C,f}$ at the quarter and mid-span points. The total complementary energy, C, of the system including the fictitious loads is then

$$C = \int_L \int_0^M d\theta\, dM - P_{B,f}\Delta_B - P_{C,f}\Delta_C - \int_0^L \Delta w\, dx \tag{i}$$

Hence

$$\frac{\partial C}{\partial P_{B,f}} = \int_L d\theta \frac{\partial M}{\partial P_{B,f}} - \Delta_B = 0 \tag{ii}$$

and

$$\frac{\partial C}{\partial P_{C,f}} = \int_L d\theta \frac{\partial M}{\partial P_{C,f}} - \Delta_C = 0 \tag{iii}$$

Assuming a linearly elastic beam Eqs. (ii) and (iii) become

$$\Delta_B = \frac{1}{EI}\int_0^L M \frac{\partial M}{\partial P_{B,f}} dx \tag{iv}$$

and

$$\Delta_C = \frac{1}{EI}\int_0^L M \frac{\partial M}{\partial P_{C,f}} dx \tag{v}$$

From A to B

$$M = \left(\frac{3}{4}P_{B,f} + \frac{1}{2}P_{C,f} + \frac{wL}{2}\right)x - \frac{wx^2}{2}$$

so that

$$\frac{\partial M}{\partial P_{B,f}} = \frac{3}{4}x, \quad \frac{\partial M}{\partial P_{C,f}} = \frac{1}{2}x$$

From B to C

$$M = \left(\frac{3}{4}P_{B,f} + \frac{1}{2}P_{C,f} + \frac{wL}{2}\right)x - \frac{wx^2}{2} - P_{B,f}\left(x - \frac{L}{4}\right)$$

giving

$$\frac{\partial M}{\partial P_{B,f}} = \frac{1}{4}(L-x), \quad \frac{\partial M}{\partial P_{C,f}} = \frac{1}{2}x$$

From C to D

$$M = \left(\frac{1}{4}P_{B,f} + \frac{1}{2}P_{C,f} + \frac{wL}{2}\right)(L-x) - \frac{w}{2}(L-x)^2$$

so that

$$\frac{\partial M}{\partial P_{B,f}} = \frac{1}{4}(L-x), \quad \frac{\partial M}{\partial P_{C,f}} = \frac{1}{2}(L-x)$$

Substituting these values in Eqs. (iv) and (v) and remembering that $P_{B,f} = P_{C,f} = 0$, we have, from Eq. (iv)

$$\Delta_B = \frac{1}{EI}\left\{\int_0^{L/4}\left(\frac{wLx}{2} - \frac{wx^2}{2}\right)\frac{3x}{4}dx + \int_{L/4}^{L/2}\left(\frac{wLx}{2} - \frac{wx^2}{2}\right)\frac{1}{4}(L-x)\,dx\right.$$
$$\left. + \int_{L/2}^{L}\left(\frac{wLx}{2} - \frac{wx^2}{2}\right)\frac{1}{4}(L-x)\,dx\right\}$$

from which

$$\Delta_B = \frac{57wL^4}{6144EI}$$

Similarly

$$\Delta_C = \frac{5wL^4}{384EI}$$

which is the result obtained by the double integration method in Ex. 13.4.

EXAMPLE 15.16

Use the principle of the stationary value of the total complementary energy of a system to calculate the horizontal displacement of the point C in the frame of Ex. 15.10.

Referring to Fig. 15.16 we apply a horizontal fictitious load, $P_{C,f}$, at C. The total complementary energy of the system, including the fictitious load, is given by

$$C = \int_L \int_0^M d\theta\, dM - P_{C,f}\Delta_{C,h} - \int_0^{2L} \Delta w dx - W\Delta_{C,h} \tag{i}$$

where $\Delta_{C,h}$ is the actual horizontal displacement of C. Then

$$\frac{\partial C}{\partial P_{C,f}} = \int_L d\theta \frac{\partial M}{\partial P_{C,f}} - \Delta_{C,h} = 0 \tag{ii}$$

Then

$$\Delta_{C,h} = \frac{1}{EI}\int_0^{2L} M\frac{\partial M}{\partial P_{C,f}}\,dx + \frac{1}{EI}\int_0^{L} M\frac{\partial M}{\partial P_{C,f}}\,dy \qquad \text{(iii)}$$

In AB

$$M = (W + P_{C,f})L - (wx^2/2)$$

so that

$$\frac{\partial M}{\partial P_{C,f}} = L$$

In BC

$$M = (W + P_{C,f})y$$

so that

$$\frac{\partial M}{\partial P_{C,f}} = y$$

Substituting these expressions in Eq. (iii) and remembering that $P_{C,f} = 0$ we have

$$\Delta_{C,h} = \frac{1}{2EI}\int_0^{2L}[WL - (wx^2/2)]L\,dx + \frac{1}{EI}\int_0^{L} Wy^2\,dy \qquad \text{(iv)}$$

Integrating Eq. (iv) and substituting the limits gives

$$\Delta_{C,h} = (2L^3/3EI)(2W - wL)$$

which is the solution produced in Ex. 15.10.

'emperature effects

:he principle of the stationary value of the total complementary energy in conjunction with the unit load nethod may be used to determine the effect of a temperature gradient through the depth of a beam.

Normally, if a structural member is subjected to a uniform temperature rise, t, it will expand as hown in Fig. 15.24. However, a variation in temperature through the depth of the member such as he linear variation shown in Fig. 15.25(b) causes the upper fibres to expand more than the lower ones o that bending strains, without bending stresses, are induced as shown in Fig. 15.25(a). Note that the ındersurface of the member is unstrained since the change in temperature in this region is zero.

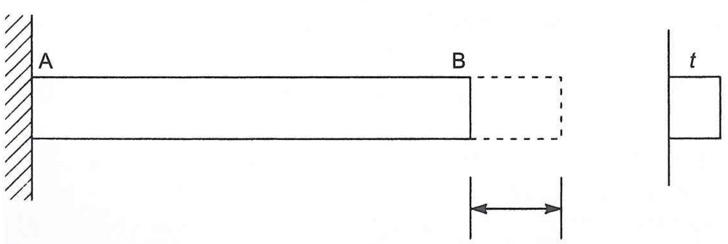


FIGURE 15.24

Expansion of a member due to

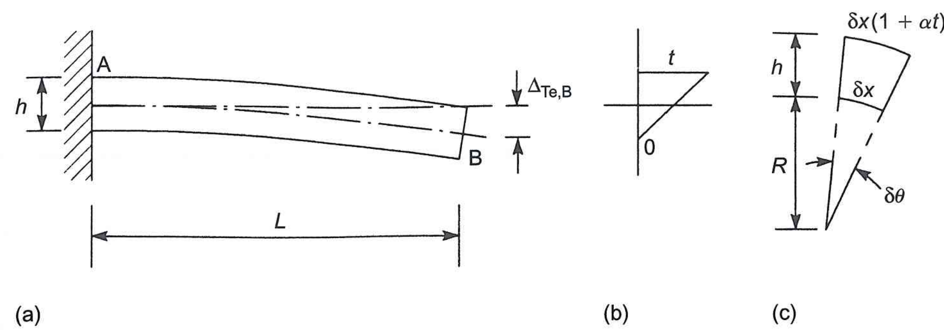


FIGURE 15.25

Bending of a beam due to a linear temperature gradient.

Consider an element, δx, of the member. The upper surface will increase in length to $\delta x(1 + \alpha t)$, while the length of the lower surface remains equal to δx as shown in Fig. 15.25(c); α is the coefficient of linear expansion of the material of the member. Thus, from Fig. 15.25(c)

$$\frac{R}{\delta x} = \frac{R + h}{\delta x(1 + \alpha t)}$$

so that

$$R = \frac{h}{\alpha t}$$

Also

$$\delta\theta = \frac{\delta x}{R}$$

whence

$$\delta\theta = \frac{\alpha t\,\delta x}{h} \qquad (15.41)$$

If we require the deflection, $\Delta_{Te,B}$, of the free end of the member due to the temperature rise, we can employ the unit load method as in Ex. 15.14. Thus, by comparison with Eq. (ii) in Ex. 15.14

$$\Delta_{Te,B} = \int_0^L d\theta\frac{\partial M}{\partial P_f} \qquad (15.42)$$

in which, as we have seen, $\partial M/\partial P_f = M_1$, the bending moment at any section of the member produced by a unit load acting vertically downwards at B. Now substituting for $\delta\theta$ in Eq. (15.42) from Eq. (15.41)

$$\Delta_{Te,B} = -\int_0^L M_1\frac{\alpha t}{h}\,dx \qquad (15.43)$$

In the case of a beam carrying actual external loads the total deflection is, from the principle of superposition (Section 3.7), the sum of the bending, shear (unless neglected) and temperature deflections. Note that in Eq. (15.43) t can vary arbitrarily along the length of the beam but only linearly with depth. Note also that the temperature gradient shown in Fig. 15.25(b) produces a hogging deflected shape for the member. Thus, strictly speaking, the radius of curvature, R, in the derivation of Eq. (15.41) is negative (compare with Fig. 9.4) so that we must insert a minus sign in Eq. (15.43) as shown.

EXAMPLE 15.17

Determine the deflection of the free end of the cantilever beam in Fig. 15.26 when subjected to the temperature gradients shown.

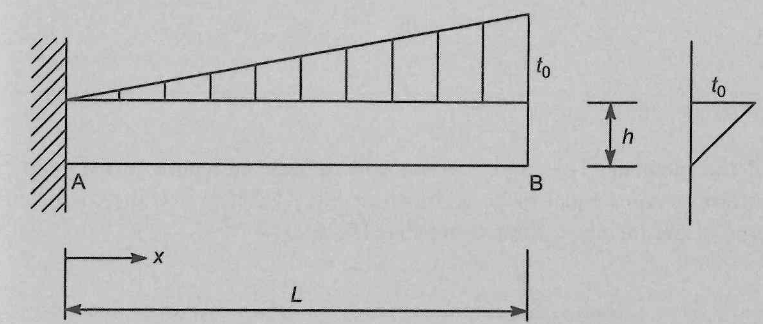


FIGURE 15.26

Deflection of a cantilever beam having linear lengthwise and depthwise temperature gradients.

The temperature, t, at any section x of the beam is given by

$$t = \frac{x}{L} t_0$$

Thus, substituting for t in Eq. (15.43), which applies since the variation of temperature through the depth of the beam is identical to that in Fig. 15.25(b), and noting that $M_1 = -1(L-x)$ we have

$$\Delta_{\mathrm{Te,B}} = -\int_0^L [-1(L-x)]\frac{\alpha}{h}\frac{x}{L}t_0 \mathrm{d}x$$

which simplifies to

$$\Delta_{\mathrm{Te,B}} = \frac{\alpha t_0}{hL}\int_0^L (Lx - x^2)\,\mathrm{d}x$$

whence

$$\Delta_{\mathrm{Te,B}} = \frac{\alpha t_0 L^2}{6h}$$

Potential energy

In the spring–mass system shown in its unstrained position in Fig. 15.27(a) the *potential energy* of the mass, m, is defined as the product of its weight and its height, h, above some arbitrary fixed datum. In other words, it possesses energy by virtue of its position. If the mass is allowed to move to the equilibrium position shown in Fig. 15.27(b) it has lost an amount of potential energy $mg\,\Delta_{\mathrm{F}}$. Thus, deflection is associated with a loss of potential energy or, alternatively, we could say that the loss of potential energy of the mass represents a *negative gain* in potential energy. Thus, if we define the potential energy of the mass as zero in its undeflected position in Fig. 15.27(a), which is the same as taking the position

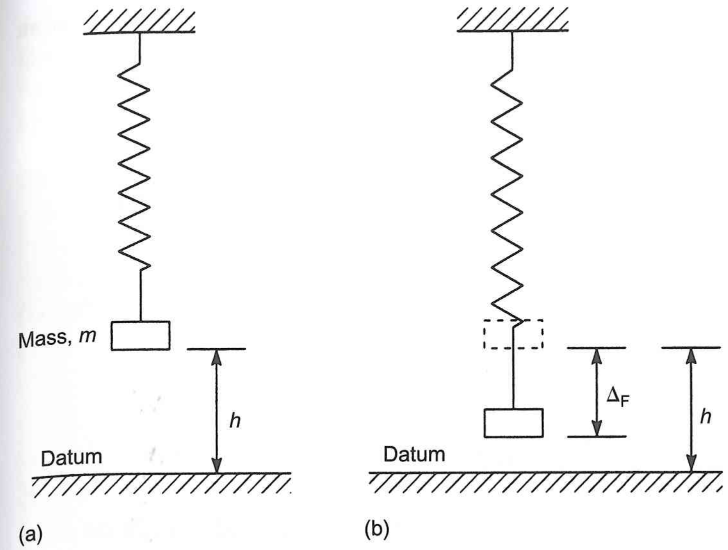


FIGURE 15.27

Potential energy of a spring–mass system.

of the datum such that $h = 0$, its actual potential energy in its deflected state in Fig. 15.27(b) is $-mgh$. In the deflected state, the total energy of the spring-mass system is the sum of the potential energy of the mass ($-mgh$) and the strain energy of the spring.

Applying the above argument to the elastic member in Fig. 15.18(a) and defining the '*total potential energy*' (TPE) of the member as the sum of the strain energy, U, of the member and the potential energy, V, of the load, we have

$$\mathrm{TPE} = U + V = \int_0^{\Delta_{j,\mathrm{F}}} P_j \mathrm{d}\Delta_j - P_{j,\mathrm{F}}\Delta_{j,\mathrm{F}} \quad \text{(see Eq. (15.24))} \tag{15.44}$$

Thus, for a structure comprising n members and subjected to a system of loads, $P_1, P_2, \ldots, P_k, \ldots, P_r$, the TPE is given by

$$\mathrm{TPE} = U + V = \sum_{j=1}^{n}\int_0^{\Delta_{j,\mathrm{F}}} P_j \mathrm{d}\Delta_j - \sum_{k=1}^{r} P_k\Delta_k \tag{15.45}$$

in which P_j is the internal force in the jth member, $\Delta_{j,\mathrm{F}}$ is its extension or contraction and Δ_k is the displacement of the load, P_k, in its line of action.

The principle of the stationary value of the total potential energy

Let us now consider an elastic body in equilibrium under a series of loads, $P_1, P_2, \ldots, P_k, \ldots, P_r$, and let us suppose that we impose infinitesimally small virtual displacements, $\delta\Delta_1, \delta\Delta_2, \ldots, \delta\Delta_k, \ldots, \delta\Delta_r$, at the points of application and in the directions of the loads. The virtual work done by the loads is then

$$\delta W_{\mathrm{e}} = \sum_{k=1}^{r} P_k \delta\Delta_k \tag{15.46}$$

This virtual work will be accompanied by an increment of virtual strain energy, δU, or internal viraccompanying virtual strains in the body itself. Therefore, from the principle of virtual work (Eq. (15.23)) we have

$$\delta W_e = \delta U$$

or

$$\delta U - \delta W_e = 0$$

Substituting for δW_e from Eq. (15.46) we obtain

$$\delta U - \sum_{k=1}^{r} P_k \delta \Delta_k = 0 \tag{15.47}$$

which may be written in the form

$$\delta\left(U - \sum_{k=1}^{r} P_k \Delta_k\right) = 0$$

in which we see that the second term is the potential energy, V, of the applied loads. Hence the equation becomes

$$\delta(U + V) = 0 \tag{15.48}$$

and we see that the TPE of an elastic system has a stationary value for all small displacements if the system is in equilibrium.

It may also be shown that if the stationary value is a minimum, the equilibrium is stable. This may be demonstrated by examining the states of equilibrium of the particle at the positions A, B and C in Fig. 15.28. The TPE of the particle is proportional to its height, h, above some arbitrary datum, u; note that a single particle does not possess strain energy, so that in this case TPE = V. Clearly, at each position of the particle, the first-order variation, $\partial(U+V)/\partial u$, is zero (indicating equilibrium) but only at B, where the TPE is a minimum, is the equilibrium stable; at A the equilibrium is unstable while at C the equilibrium is neutral.

The *principle of the stationary value of the* TPE may therefore be stated as:

The TPE of an elastic system has a stationary value for all small displacements when the system is in equilibrium; further, the equilibrium is stable if the stationary value is a minimum.

Potential energy can often be used in the approximate analysis of structures in cases where an exact analysis does not exist. We shall illustrate such an application for a simple beam in Ex. 15.18 below

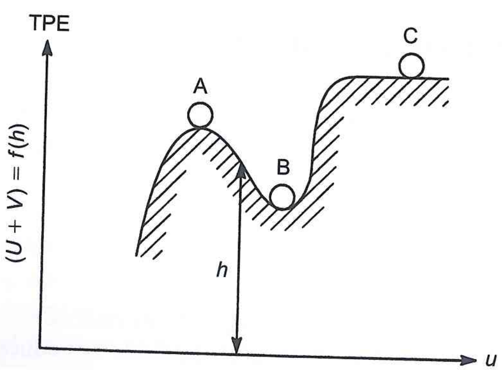


FIGURE 15.28

States of equilibrium of a parti...

and in Chapter 21 in the case of a buckled column; in both cases we shall suppose that the deflected form is unknown and has to be initially assumed (this approach is called the *Rayleigh–Ritz method*). For a linearly elastic system, of course, the methods of complementary energy and potential energy are identical.

EXAMPLE 15.18

Determine the deflection of the mid-span point of the linearly elastic, simply supported beam ABC shown in Fig. 15.29(a).

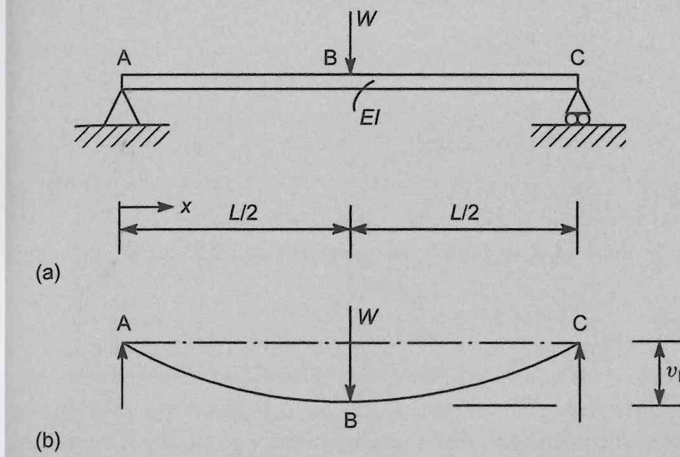


FIGURE 15.29

Approximate value for beam deflection using TPE.

We shall suppose that the deflected shape of the beam is unknown. Initially, therefore, we shall assume a deflected shape that satisfies the boundary conditions for the beam. Generally, trigonometric or polynomial functions have been found to be the most convenient where the simpler the function the less accurate the solution. Let us suppose that the displaced shape of the beam is given by

$$v = v_B \sin\frac{\pi x}{L} \tag{i}$$

in which v_B is the deflection at the mid-span point. From Eq. (i) we see that when $x = 0$ and $x = L$, $v = 0$ and that when $x = L/2$, $v = v_B$. Furthermore, $dv/dx = (\pi/L)v_B \cos(\pi x/L)$ which is zero when $x = L/2$. Thus the displacement function satisfies the boundary conditions of the beam.

The strain energy due to bending of the beam is given by Eq. (9.21), i.e.

$$U = \int_0^L \frac{M^2}{2EI}\,dx \tag{ii}$$

Also, from Eq. (13.3)

$$M = EI\frac{d^2v}{dx^2} \tag{iii}$$

Substituting in Eq. (iii) for v from Eq. (i), and for M in Eq. (ii) from Eq. (iii), we have

$$U = \frac{EI}{2}\int_0^L \frac{v_B^2\pi^4}{L^4}\sin^2\frac{\pi x}{L}\,dx$$

which gives

$$U = \frac{\pi^4 EI v_B^2}{4L^3}$$

The TPE of the beam is then given by

$$\text{TPE} = U + V = \frac{\pi^4 EI v_B^2}{4L^3} - W v_B$$

Hence, from the principle of the stationary value of the TPE

$$\frac{\partial(U+V)}{\partial v_B} = \frac{\pi^4 EI v_B}{2L^3} - W = 0$$

whence

$$v_B = \frac{2WL^3}{\pi^4 EI} = 0.02053\frac{WL^3}{EI} \tag{iv}$$

The exact expression for the deflection at the mid-span point was found in Ex. 13.5 and is

$$v_B = \frac{WL^3}{48EI} = 0.02083\frac{WL^3}{EI} \tag{v}$$

Comparing the exact and approximate results we see that the difference is less than 2%. Furthermore, the approximate deflection is less than the exact deflection because, by assuming a deflected shape, we have, in effect, forced the beam into that shape by imposing restraints; the beam is therefore stiffer.

15.4 Reciprocal theorems

There are two reciprocal theorems: one, attributed to Maxwell, is the *theorem of reciprocal displacements* (often referred to as Maxwell's reciprocal theorem) and the other, derived by Betti and Rayleigh, is the *theorem of reciprocal work*. We shall see, in fact, that the former is a special case of the latter. We shall also see that their proofs rely on the principle of superposition (Section 3.7) so that their application is limited to linearly elastic structures.

Theorem of reciprocal displacements

In a linearly elastic body a load, P_1, applied at a point 1 will produce a displacement, Δ_1, at the point and in its own line of action given by

$$\Delta_1 = a_{11}P_1$$

in which a_{11} is a *flexibility coefficient* which is defined as the displacement at the point 1 in the direction of P_1 produced by a unit load at the point 1 in the direction of P_1. It follows that if the elastic body is subjected to a series of loads, P_1, P_2, ..., P_k, ..., P_r, each of the loads will contribute to the displacement of point 1. Thus the corresponding displacement, Δ_1, at the point 1 (i.e. the total displacement in the direction of P_1 produced by all the loads) is then

$$\Delta_1 = a_{11}P_1 + a_{12}P_2 + \ldots + a_{1k}P_k + \ldots + a_{1r}P_r$$

in which a_{12} is the displacement at the point 1 in the direction of P_1 produced by a unit load at 2 in the direction of P_2, and so on. The corresponding displacements at the points of application of the loads are then

$$\left.\begin{aligned} \Delta_1 &= a_{11}P_1 + a_{12}P_2 + \ldots + a_{1k}P_k + \ldots + a_{1r}P_r \\ \Delta_2 &= a_{21}P_1 + a_{22}P_2 + \ldots + a_{2k}P_k + \ldots + a_{2r}P_r \\ &\vdots \\ \Delta_k &= a_{k1}P_1 + a_{k2}P_2 + \ldots + a_{kk}P_k + \ldots + a_{kr}P_r \\ &\vdots \\ \Delta_r &= a_{r1}P_1 + a_{r2}P_2 + \ldots + a_{rk}P_k + \ldots + a_{rr}P_r \end{aligned}\right\} \tag{15.49}$$

or, in matrix form

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_k \\ \vdots \\ \Delta_r \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1k} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2k} & \cdots & a_{2r} \\ & & \vdots & & & \\ a_{k1} & a_{k2} & \cdots & a_{kk} & \cdots & a_{kr} \\ & & \vdots & & & \\ a_{r1} & a_{r2} & \cdots & a_{rk} & \cdots & a_{rr} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_k \\ \vdots \\ P_r \end{Bmatrix} \tag{15.50}$$

which may be written in matrix shorthand notation as

$$\{\Delta\} = [A]\{P\}$$

Suppose now that a linearly elastic body is subjected to a gradually applied load, P_1, at a point 1 and then, while P_1 remains in position, a load P_2 is gradually applied at another point 2. The total strain energy, U_1, of the body is equal to the external work done by the loads; thus

$$U_1 = \frac{P_1}{2}(a_{11}P_1) + \frac{P_2}{2}(a_{22}P_2) + P_1(a_{12}P_2) \tag{15.51}$$

The third term on the right-hand side of Eq. (15.51) results from the additional work done by P_1 as it is displaced through a further distance $a_{12}P_2$ by the action of P_2. If we now remove the loads and then apply P_2 followed by P_1, the strain energy, U_2, is given by

$$U_2 = \frac{P_2}{2}(a_{22}P_2) + \frac{P_1}{2}(a_{11}P_1) + P_2(a_{21}P_1) \tag{15.52}$$

By the principle of superposition the strain energy of the body is independent of the order in which the loads are applied. Hence

$$U_1 = U_2$$

so that

$$a_{12} = a_{21} \tag{15.53}$$

Thus, in its simplest form, the theorem of reciprocal displacements states that:

The displacement at a point 1 in a given direction due to a unit load at a point 2 in a second direction is equal to the displacement at the point 2 in the second direction due to a unit load at the point 1 in the given direction.

The theorem of reciprocal displacements may also be expressed in terms of moments and rotations. Thus:

The rotation at a point 1 due to a unit moment at a point 2 is equal to the rotation at the point 2 pro-

Finally we have:

The rotation in radians at a point 1 due to a unit load at a point 2 is numerically equal to the displacement at the point 2 in the direction of the unit load due to a unit moment at the point 1.

EXAMPLE 15.19

A cantilever 800 mm long with a prop 500 mm from its built-in end deflects in accordance with the following observations when a concentrated load of 40 kN is applied at its free end:

Distance from fixed end (mm)	0	100	200	300	400	500	600	700	800
Deflection (mm)	0	0.3	1.4	2.5	1.9	0	−2.3	−4.8	−10.6

What will be the angular rotation of the beam at the prop due to a 30 kN load applied 200 mm from the built-in end together with a 10 kN load applied 350 mm from the built-in end?

The initial deflected shape of the cantilever is plotted to a suitable scale from the above observations and is shown in Fig. 15.30(a). Thus, from Fig. 15.30(a) we see that the deflection at D due to a 40 kN load at C is 1.4 mm. Hence the deflection at C due to a 40 kN load at D is, from the reciprocal theorem, 1.4 mm. It follows that the deflection at C due to a 30 kN load at D is equal to $(3/4) \times (1.4) = 1.05$ mm. Again, from Fig. 15.30(a), the deflection at E due to a 40 kN load at C is 2.4 mm. Thus the deflection at C due to a 10 kN load at E is equal to $(1/4) \times (2.4) = 0.6$ mm. Therefore the total deflection at C due to a 30 kN load at D and a 10 kN load at E is $1.05 + 0.6 = 1.65$ mm. From Fig. 15.30(b) we see that the rotation of the beam at B is given by

$$\theta_B = \tan^{-1}\left(\frac{1.65}{300}\right) = \tan^{-1}(0.0055)$$

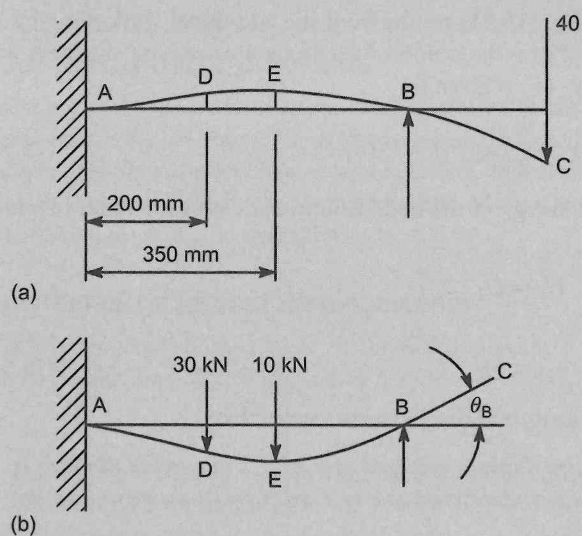


FIGURE 15.30

Deflection of a propped cantilever using the reciprocal theorem.

or

$$\theta_B = 0°19'$$

EXAMPLE 15.20

An elastic member is pinned to a drawing board at its ends A and B. When a moment, M, is applied at A, A rotates by θ_A, B rotates by θ_B and the centre deflects by δ_1. The same moment, M, applied at B rotates B by θ_C and deflects the centre through δ_2. Find the moment induced at A when a load, W, is applied to the centre in the direction of the measured deflections, and A and B are restrained against rotation.

The three load conditions and the relevant displacements are shown in Fig. 15.31. Thus, from Fig. 15.31(a) and (b) the rotation at A due to M at B is, from the reciprocal theorem, equal to the rotation at B due to M at A.

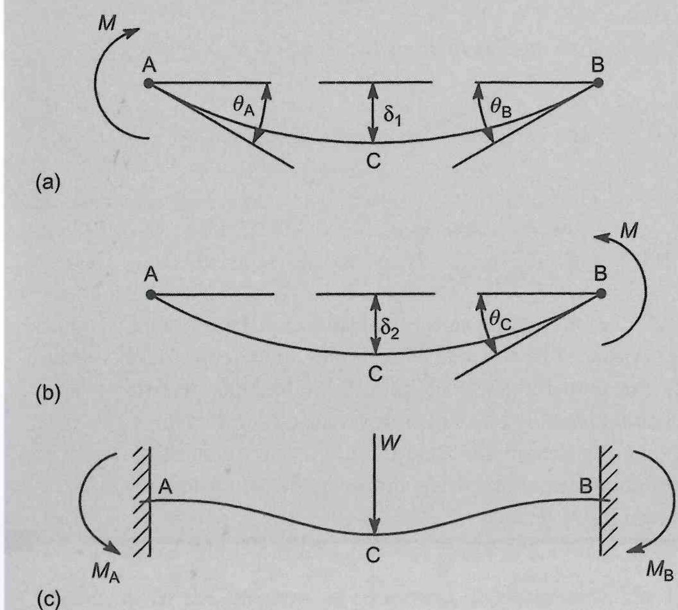


FIGURE 15.31

Model analysis of a fixed beam.

Thus

$$\theta_{A(b)} = \theta_B$$

It follows that the rotation at A due to M_B at B is

$$\theta_{A(c),1} = \frac{M_B}{M}\theta_B \quad \text{(i)}$$

where (b) and (c) refer to (b) and (c) in Fig. 15.31.

Also, the rotation at A due to a unit load at C is equal to the deflection at C due to a unit moment at A. Therefore

$$\frac{\theta_{A(c),2}}{W} = \frac{\delta_1}{M}$$

or

$$\theta_{A(c),2} = \frac{W}{M}\delta_1 \quad \text{(ii)}$$

in which $\theta_{A(c),2}$ is the rotation at A due to W at C. Finally the rotation at A due to M_A at A is, from Fig. 15.31(a) and (c)

$$\theta_{A(c),3} = \frac{M_A}{M}\theta_A \quad \text{(iii)}$$

The total rotation at A produced by M_A at A, W at C and M_B at B is, from Eqs (i), (ii) and (iii)

$$\theta_{A(c),1} + \theta_{A(c),2} + \theta_{A(c),3} = \frac{M_B}{M}\theta_B + \frac{W}{M}\delta_1 + \frac{M_A}{M}\theta_A = 0 \quad \text{(iv)}$$

since the end A is restrained against rotation. In a similar manner the rotation at B is given by

$$\frac{M_B}{M}\theta_C + \frac{W}{M}\delta_2 + \frac{M_A}{M}\theta_B = 0 \quad \text{(v)}$$

Solving Eqs (iv) and (v) for M_A gives

$$M_A = W\left(\frac{\delta_2\theta_B - \delta_1\theta_C}{\theta_A\theta_C - \theta_B^2}\right)$$

The fact that the arbitrary moment, M, does not appear in the expression for the restraining moment at A (similarly it does not appear in M_B) produced by the load W indicates an extremely useful application of the reciprocal theorem, namely the model analysis of statically indeterminate structures. For example, the fixed beam of Fig. 15.31(c) could possibly be a full-scale bridge girder. It is then only necessary to construct a model, say, of perspex, having the same flexural rigidity, EI, as the full-scale beam and measure rotations and displacements produced by an arbitrary moment, M, to obtain the fixed-end moments in the full-scale beam supporting a full-scale load.

Theorem of reciprocal work

Let us suppose that a linearly elastic body is to be subjected to two systems of loads, $P_1, P_2, \ldots, P_k, \ldots, P_r$ and, $Q_1, Q_2, \ldots, Q_i, \ldots, Q_m$, which may be applied simultaneously or separately. Let us also suppose that corresponding displacements are $\Delta_{P,1}, \Delta_{P,2}, \ldots, \Delta_{P,k}, \ldots, \Delta_{P,r}$ due to the loading system, P, and $\Delta_{Q,1}, \Delta_{Q,2}, \ldots, \Delta_{Q,i}, \ldots, \Delta_{Q,m}$ due to the loading system, Q. Finally, let us suppose that the loads, P, produce displacements $\Delta'_{Q,1}, \Delta'_{Q,2}, \ldots, \Delta'_{Q,i}, \ldots, \Delta'_{Q,m}$ at the points of application and in the direction of the loads, Q, while the loads, Q, produce displacements $\Delta'_{P,1}, \Delta'_{P,2}, \ldots, \Delta'_{P,k}, \ldots, \Delta'_{P,r}$ at the points of application and in the directions of the loads, P.

Now suppose that the loads P and Q are applied to the elastic body gradually and simultaneously. The total work done, and hence the strain energy stored, is then given by

$$\begin{aligned} U_1 = {} & \frac{1}{2}P_1(\Delta_{P,1} + \Delta'_{P,1}) + \frac{1}{2}P_2(\Delta_{P,2} + \Delta'_{P,2}) + \ldots + \frac{1}{2}P_K(\Delta_{P,k} + \Delta'_{P,k}) \\ & + \ldots + \frac{1}{2}P_r(\Delta_{P,r} + \Delta'_{P,r}) + \frac{1}{2}Q_1(\Delta_{Q,1} + \Delta'_{Q,1}) + \frac{1}{2}Q_2(\Delta_{Q,2} + \Delta'_{Q,2}) \\ & + \ldots + \frac{1}{2}Q_i(\Delta_{Q,i} + \Delta'_{Q,i}) + \ldots + \frac{1}{2}Q_m(\Delta_{Q,m} + \Delta'_{Q,m}) \end{aligned} \quad (15.54)$$

If now we apply the P-loading system followed by the Q-loading system, the total strain energy stored is

$$\begin{aligned} U_2 = {} & \frac{1}{2}P_1\Delta_{P,1} + \frac{1}{2}P_2\Delta_{P,2} + \ldots + \frac{1}{2}P_k\Delta_{P,k} + \ldots + \frac{1}{2}P_r\Delta_{P,r} + \frac{1}{2}Q_1\Delta_{Q,1} + \frac{1}{2}Q_2\Delta_{Q,2} \\ & + \ldots + \frac{1}{2}Q_i\Delta_{Q,i} + \ldots + \frac{1}{2}Q_m\Delta_{Q,m} + P_1\Delta'_{P,1} + P_2\Delta'_{P,2} + P_k\Delta'_{P,k} + \ldots + P_r\Delta'_{P,r} \end{aligned} \quad (15.55)$$

Since, by the principle of superposition, the total strain energies, U_1 and U_2, must be the same, we have from Eqs (15.54) and (15.55)

$$\begin{aligned} & -\frac{1}{2}P_1\Delta'_{P,1} - \frac{1}{2}P_2\Delta'_{P,2} - \ldots - \frac{1}{2}P_k\Delta'_{P,k} - \ldots - \frac{1}{2}P_r\Delta'_{P,r} \\ & = -\frac{1}{2}Q_1\Delta'_{Q,1} - \frac{1}{2}Q_2\Delta'_{Q,2} - \ldots - \frac{1}{2}Q_i\Delta'_{Q,i} - \ldots - \frac{1}{2}Q_m\Delta'_{Q,m} \end{aligned}$$

In other words

$$\sum_{k=1}^{r} P_k\Delta'_{P,k} = \sum_{i=1}^{m} Q_m\Delta'_{Q,m} \quad (15.56)$$

The expression on the left-hand side of Eq. (15.56) is the sum of the products of the P loads and their corresponding displacements produced by the Q loads. The right-hand side of Eq. (15.56) is the sum of the products of the Q loads and their corresponding displacements produced by the P loads. Thus the *theorem of reciprocal work* may be stated as:

The work done by a first loading system when moving through the corresponding displacements produced by a second loading system is equal to the work done by the second loading system when moving through the corresponding displacements produced by the first loading system.

Again, as in the theorem of reciprocal displacements, the loading systems may be either forces or moments and the displacements may be deflections or rotations.

If, in the above, the P- and Q-loading systems comprise just two loads, say P_1 and Q_2, then, from Eq. (15.56), we see that

$$P_1(a_{12}Q_2) = Q_2(a_{21}P_1)$$

so that

$$a_{12} = a_{21}$$

as in the theorem of reciprocal displacements. Therefore, as stated initially, we see that the theorem of reciprocal displacements is a special case of the theorem of reciprocal work.

In addition to the use of the reciprocal theorems in the model analysis of structures as described in Ex. 15.20, they are used to establish the symmetry of, say, the stiffness matrix in the matrix analysis of some structural systems. We shall examine this procedure in Chapter 16.

PROBLEMS

P.15.1 Use the principle of virtual work to determine the support reactions in the beam ABCD shown in Fig. P.15.1.

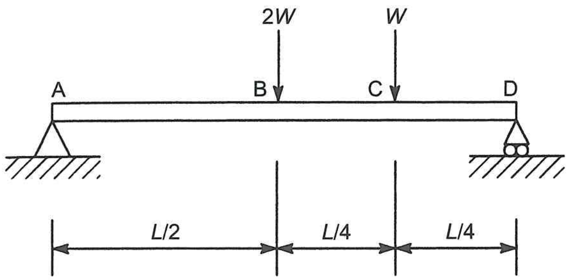


FIGURE P.15.1

P.15.2 Find the support reactions in the beam ABC shown in Fig. P.15.2 using the principle of virtual work.

Ans. $R_A = (W + 2wL)/4$ $R_C = (3W + 2wL)/4$.

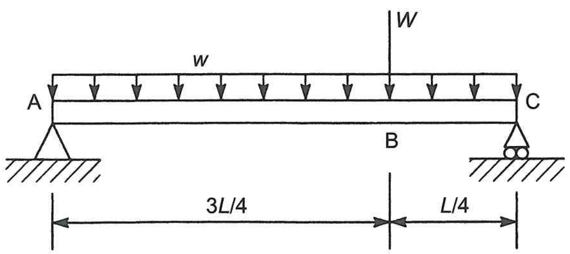


FIGURE P.15.2

P.15.3 Determine the reactions at the built-in end of the cantilever beam ABC shown in Fig. P.15.3 using the principle of virtual work.

Ans. $R_A = 3W$ $M_A = 2.5WL$.

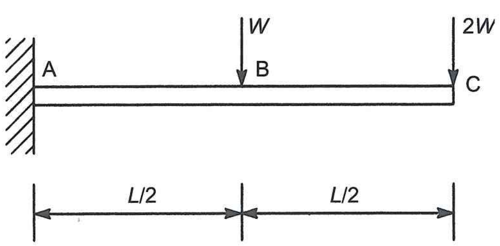


FIGURE P.15.3

P.15.4 Find the bending moment at the three-quarter-span point in the beam shown in Fig. P.15.4. Use the principle of virtual work.

Ans. $3wL^2/32$.

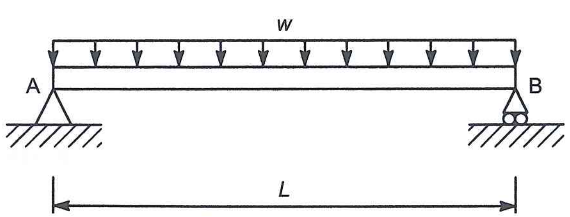


FIGURE P.15.4

P.15.5 Calculate the forces in the members FG, GD and CD of the truss shown in Fig. P.15.5 using the principle of virtual work. All horizontal and vertical members are 1 m long.

Ans. FG = +20 kN GD = +28.3 kN CD = −20 kN.

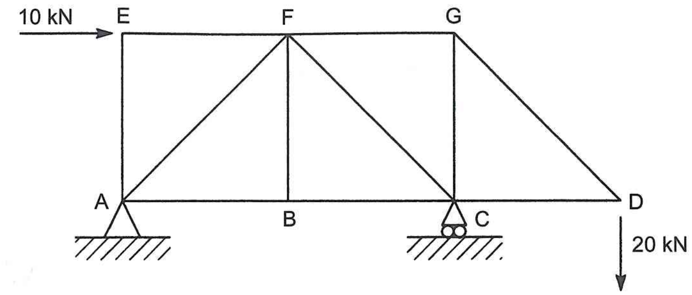


FIGURE P.15.5

P.15.6 The frame shown in Fig. P.15.6 consists of a cranked beam simply supported at A and F and reinforced by a tie bar pinned to the beam at B and E. The beam carries a uniformly distributed load of intensity, w, over the outer parts AB and EF. Considering the effects of bending and axial load determine the axial force in the tie bar and the bending moments at B and C. The bending stiffness of the beam is EI and its cross sectional area is $3A$ while the corresponding values for the tie bar are EI and A.

Ans. Force in tie bar $T = 9wL^3/8[L^2 + (4I/A)]$. M (at B) $= wL^2$ (anticlockwise), M (at C) $= (TL - wL^2)/2$ (clockwise).

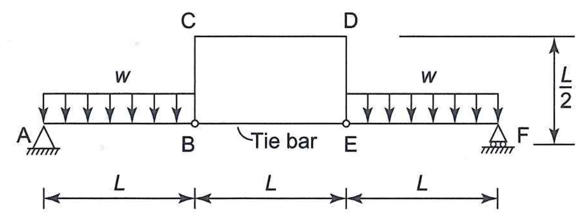


FIGURE P.15.6

P.15.7 The flat tension spring shown in Fig. P.15.7 consists of a length of wire of circular cross section, having a diameter, d, and Young's modulus, E. The spring consists of n open loops each of which subtends an angle of $3\pi/2$ radians at its centre; the length between the ends of the spring is L. Considering bending and axial strains only calculate the stiffness of the spring.

Ans. $(\sqrt{2})\ \pi E d^4/L\{[48L^2(\pi + 1)/n^2 d^2] + (3\pi - 2)\}$.

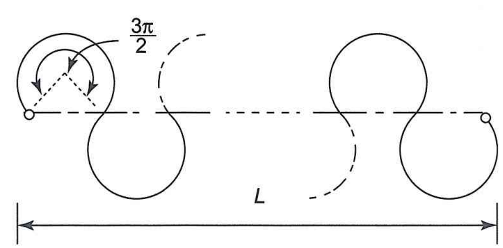


FIGURE P.15.7

P.15.8 Use the unit load method to calculate the vertical displacements at the quarter and mid-span points in the beam shown in Fig. P.15.8.